

The University of Chicago

THE EQUILIBRIUM OF A PERFECT
COMPRESSIBLE FLUID
CONFIGURATION

A DISSERTATION SUBMITTED TO THE FACULTY OF THE
DIVISION OF THE PHYSICAL SCIENCES IN CANDIDACY
FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF ASTRONOMY AND ASTROPHYSICS
1942

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WASLEY KROGDAHL

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ABSTRACT

The equilibrium of a perfect compressible fluid configuration is considered, and the equations for the perturbations induced by rotational and/or tidal forces are derived. The results are obtained in a sufficiently general form to be applicable to any model and are seen to be simple generalizations of what is already known for polytropes. Numerical application is made to three nonpolytropic models.

It is proposed to consider the problem of determining the distortion of a perfect compressible fluid configuration which is subject to the perturbing effort of a slow rotation, or a weak tidal field, separately or jointly. The distortions obviously must be functions of the pressure and density distributions which would describe the configuration in gravitational equilibrium in the absence of the perturbing forces. A determination of the specific functional dependence of the distortions on the structure of the associated unperturbed configuration is the problem set forth for solution.

The problem has been solved in detail¹ for the case in which the unperturbed configuration is a polytrope. In particular, it has been shown that the ellipticity of a configuration of given mass and mean density rotating with constant angular velocity depends only on the polytropic index n . Conversely, from an observed ellipticity one can always infer an "effective polytropic index." It is this circumstance which underlies the conclusions that have been drawn² recently concerning the internal distribution of matter in stars whose rotation of the line of apsides can be observed. While the determination of such effective polytropic indices is of value in giving a general insight into the problem, it is clear that one is here dealing with a dependence (of ellipticity) on the total structure of a configuration; this is equivalent to a dependence on a single parameter n only in the very idealized case in which the configuration is a polytrope. Therefore, if one were to have reliable knowledge concerning the density and pressure distributions in normal static nonpolytropic stars, it would be desirable to know how such a configuration would be distorted. This is the problem whose solution is to be attempted. The general method follows closely those developed by Milne, von Zeipel, and Chandrasekhar in their investigations on the distortion of polytropic distributions.

Consider a nonviscous compressible fluid mass at rest in an otherwise empty space. In gravitational equilibrium the mass will assume a spherical form, and its structure can be specified by a statement of its density distribution function $\rho = \rho(r)$ (where r is the radial distance from the center of the sphere). In addition, since the fluid is compressible, there must exist, by definition, a pressure-density relation,

$$P = P(\rho), \quad (1)$$

by which the pressure P at any point is related to the density ρ at that point. Further, the pressure and density must satisfy the hydrodynamical equations

$$\frac{1}{\rho} \frac{\partial P}{\partial x^s} = F_s = \frac{\partial \mathfrak{B}}{\partial x^s}, \quad (2)$$

¹ S. Chandrasekhar, *M.N.*, **93**, 1933: "The Equilibrium of Distorted Polytropes: I, The Rotational Problem," p. 390; "II, The Tidal Problem," p. 449; "III, The Double Star Problem," p. 462; "IV, The Rotational and Tidal Distortions as Functions of the Density Distribution," p. 539.

² T. G. Cowling, *M.N.*, **98**, 734, 1938; T. E. Sterne, *M.N.*, **99**, 451, 1939.

where the x^s 's are the co-ordinates in the space and F_s the (covariant) acceleration vector-field throughout the space, and where $\mathfrak{B} = \mathfrak{B}(r)$, the gravitational potential function, is a solution of Poisson's equation

$$\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^s} \left(\sqrt{g} g^{sq} \frac{\partial \mathfrak{B}}{\partial x^q} \right) = -4\pi G \rho, \quad (3)$$

g^{sq} being the fundamental metric tensor, g the value of the determinant $|g^{sq}|$, and G the gravitational constant. The pressure P must then satisfy a second-order ordinary differential equation in r whose solution, subject to (1), can in principle always be found.

Now let it be assumed that upon the static gravitational acceleration-field there is superposed an arbitrary additive acceleration-field F_s'' . The functions ρ and $\mathfrak{B}$ will, in general, no longer be spherically symmetrical. Letting the new (altered) gravitational acceleration-field be

$$F'_s = \frac{\partial \mathfrak{B}'}{\partial x^s}, \quad (4)$$

the hydrodynamical equations may now be written

$$\frac{1}{\rho} \frac{\partial P}{\partial x^s} = F'_s + F''_s = \frac{\partial \mathfrak{B}'}{\partial x^s} + F''_s, \quad (5)$$

which, if F''_s is derivable from a potential function $\mathfrak{B}''$, can be written

$$\frac{1}{\rho} \frac{\partial P}{\partial x^s} = \frac{\partial}{\partial x^s} (\mathfrak{B}' + \mathfrak{B}''). \quad (6)$$

Equations (1), (2), and (3) are the fundamental equations of the problem.

The difficulties which beset the problem of finding a general and exact solution of the set of the fundamental equations are twofold. The first is that of expressing the functions F''_s in terms of the variables x^s and appropriate parameters;³ it gives rise to no difficulty of principle, however. The second difficulty is the more fundamental one of obtaining explicit solutions of equations (5) in terms of the variables and any necessary parameters which may enter. This problem is to be considered in some detail.

I. A SLOWLY ROTATING CONFIGURATION WITH CONSTANT ANGULAR VELOCITY

Consider a fluid mass rotating slowly about some fixed axis, which for convenience may be taken to be the z -axis of a Cartesian co-ordinate system. The dynamical equations (6) for the rotating star are, in polar co-ordinates ($x^1 = r$, $x^2 = \mu = \cos \varphi$, $x^3 = \theta$, $\mathfrak{B}'' = \frac{1}{2}\omega^2 r^2(1 - \mu^2)$),

$$\frac{1}{\rho} \frac{\partial P}{\partial r} = \frac{\partial \mathfrak{B}'}{\partial r} + r(1 - \mu^2)\omega^2 \quad (7)$$

and

$$\frac{1}{\rho} \frac{\partial P}{\partial \mu} = \frac{\partial \mathfrak{B}'}{\partial \mu} - r^2 \mu \omega^2, \quad (8)$$

³ There is no difficulty in this respect for acceleration-fields due to the mass's rotation about a fixed axis, but the complete tidal acceleration-field induced by other bodies is a striking example of a field whose mathematical expression is not simple.

where ω is the (constant)⁴ angular velocity of rotation. Similarly, in polar co-ordinates Poisson's equation (eq. [3]) becomes

$$\frac{1}{r^2} \left\{ \frac{\partial}{\partial r} \left(r^2 \frac{\partial \mathfrak{B}'}{\partial r} \right) + \frac{\partial}{\partial \mu} \left((1 - \mu^2) \frac{\partial \mathfrak{B}'}{\partial \mu} \right) \right\} = -4\pi G \rho. \quad (9)$$

The first step in solving the problem is to transform the partial differential equations, by separation, to a set of equivalent ordinary differential equations. This process may be effected more conveniently, however, by first interposing an auxiliary step. It is to be noted that one seeks solutions of the form $\rho = \rho(r, \mu; \omega)$, $P = P(r, \mu; \omega)$, and $\mathfrak{B}' = \mathfrak{B}'(r, \mu; \omega)$. Now, equations (7), (8), and (9) contain only functions analytic in ω , and it is therefore possible to express the desired solutions in the form of power series in ω , convergent in some nonzero interval of ω . By this means, ω can be eliminated from further consideration and the problem thereby simplified.

For general purposes it will be well to have all the necessary equations expressed in terms of dimensionless variables. Therefore, let a new set of variables be defined by the relations

$$\left. \begin{aligned} r &= a \xi, & \rho &= \rho_c \zeta, \\ P &= P_c \eta = (4\pi G \rho_c^2 a^2) \eta, \\ \mathfrak{B}' &= (4\pi G \rho_c a^2) \chi'. \end{aligned} \right\} \quad (10)$$

Equations (1) and (9) then become

$$\eta = \eta(\zeta) \quad (11)$$

and

$$\frac{1}{\xi^2} \left\{ \frac{\partial}{\partial \xi} \left(\xi^2 \frac{\partial \chi'}{\partial \xi} \right) + \frac{\partial}{\partial \mu} \left((1 - \mu^2) \frac{\partial \chi'}{\partial \mu} \right) \right\} = -\zeta, \quad (12)$$

and, letting

$$\epsilon = \frac{\omega^2}{4\pi G \rho_c}, \quad (13)$$

equations (7) and (8) become

$$\frac{1}{\zeta} \frac{\partial \eta}{\partial \xi} = \frac{\partial \chi'}{\partial \xi} + \epsilon \xi (1 - \mu^2) \quad (14)$$

and

$$\frac{1}{\zeta} \frac{\partial \eta}{\partial \mu} = \frac{\partial \chi'}{\partial \mu} - \epsilon \xi^2 \mu, \quad (15)$$

respectively.

Now assume that ζ , η , and χ' can be represented by the power series

$$\zeta(\xi, \mu; \epsilon) = \zeta_0(\xi) + \epsilon \zeta_1(\xi, \mu) + \epsilon^2 \zeta_2(\xi, \mu) + \dots, \quad (16)$$

$$\eta(\xi, \mu; \epsilon) = \eta_0(\xi) + \epsilon \eta_1(\xi, \mu) + \epsilon^2 \eta_2(\xi, \mu) + \dots, \quad (17)$$

and

$$\chi'(\xi, \mu; \epsilon) = \chi'_0(\xi) + \epsilon \chi'_1(\xi, \mu) + \epsilon^2 \chi'_2(\xi, \mu) + \dots. \quad (18)$$

⁴It should be noted that, by the converse of a theorem due to Poincaré (*Leçons sur les hypothèses cosmogoniques*, p. 32, 2d ed., Paris, 1913), the constancy of the angular momentum guarantees the existence of some pressure-density relation such as (1). In other words, in the (P, ρ) -plane, the value of the increment $d\rho$ associated to an increment dP is not a function of the direction of the generating spatial displacement in the star. A like statement cannot be made for a general variable angular velocity.

Inserting these expressions into equations (12), (14), and (15) and equating the coefficients of like powers of ϵ , one finds that, to the second order in ϵ ,

$$\frac{1}{\xi^2} \left\{ \frac{d}{d\xi} \left(\xi^2 \frac{d\chi_0'}{d\xi} \right) \right\} = -\zeta_0, \quad (19)$$

$$\frac{1}{\xi^2} \left\{ \frac{\partial}{\partial \xi} \left(\xi^2 \frac{\partial \chi_1'}{\partial \xi} \right) + \frac{\partial}{\partial \mu} \left((1 - \mu^2) \frac{\partial \chi_1'}{\partial \mu} \right) \right\} = -\zeta_1, \quad (20)$$

$$\frac{1}{\xi^2} \left\{ \frac{\partial}{\partial \xi} \left(\xi^2 \frac{\partial \chi_2'}{\partial \xi} \right) + \frac{\partial}{\partial \mu} \left((1 - \mu^2) \frac{\partial \chi_2'}{\partial \mu} \right) \right\} = -\zeta_2, \quad (21)$$

$$\frac{1}{\zeta_0} \frac{d\eta_0}{d\xi} = \frac{d\chi_0'}{d\xi}, \quad (22)$$

$$\frac{1}{\zeta_0} \left\{ \frac{\partial \eta_1}{\partial \xi} - \left(\frac{\zeta_1}{\zeta_0} \right) \frac{d\eta_0}{d\xi} \right\} = \frac{\partial \chi_1'}{\partial \xi} + (1 - \mu^2) \xi, \quad (23)$$

$$\frac{1}{\zeta_0} \left\{ \frac{\partial \eta_2}{\partial \xi} - \left(\frac{\zeta_1}{\zeta_0} \right) \frac{\partial \eta_1}{\partial \xi} - \frac{d\eta_0}{d\xi} \left[\frac{\zeta_2}{\zeta_0} - \left(\frac{\zeta_1}{\zeta_0} \right)^2 \right] \right\} = \frac{\partial \chi_2'}{\partial \xi}, \quad (24)$$

$$\frac{1}{\zeta_0} \frac{\partial \eta_1}{\partial \mu} = \frac{\partial \chi_1'}{\partial \mu} - \mu \xi^2, \quad (25)$$

$$\frac{1}{\zeta_0} \left\{ \frac{\partial \eta_2}{\partial \mu} - \frac{\zeta_1}{\zeta_0} \frac{\partial \eta_1}{\partial \mu} \right\} = \frac{\partial \chi_2'}{\partial \mu}. \quad (26)$$

In addition, the relation implied by (11) must impose necessary restrictions on η and ζ . For the function $\eta(\zeta)$ must be such that

$$\eta(\zeta_0) = \eta_0 \quad (27)$$

and

$$\left. \begin{aligned} \eta(\zeta(\xi, \mu; \epsilon)) &= \eta(\zeta_0 + \epsilon \zeta_1 + \epsilon^2 \zeta_2 + \dots) \\ &= \eta(\zeta_0) + \epsilon \left(\frac{d\eta}{d\zeta} \right)_{\zeta=\zeta_0} + \frac{\epsilon^2}{2} \left(\frac{d^2\eta}{d\zeta^2} \right)_{\zeta=\zeta_0} + \dots \end{aligned} \right\} \quad (28)$$

However,

$$\left(\frac{d\eta}{d\epsilon} \right)_{\epsilon=0} = \left(\frac{d\eta}{d\zeta} \frac{d\zeta}{d\epsilon} \right)_{\epsilon=0} = \zeta_1 \frac{d\eta_0}{d\zeta_0} \quad (29)$$

and

$$\left(\frac{d^2\eta}{d\epsilon^2} \right)_{\epsilon=0} = \left[\frac{d^2\eta}{d\zeta^2} \left(\frac{d\zeta}{d\epsilon} \right)^2 + \frac{d\eta}{d\zeta} \frac{d^2\zeta}{d\epsilon^2} \right]_{\epsilon=0} = \frac{d^2\eta_0}{d\zeta_0^2} (\zeta_1)^2 + 2 \frac{d\eta_0}{d\zeta_0} \zeta_2, \quad (30)$$

so that

$$\eta(\zeta(\xi, \mu; \epsilon)) = \eta_0 + \epsilon \left[\zeta_1 \frac{d\eta_0}{d\zeta_0} \right] + \frac{\epsilon^2}{2} \left[(\zeta_1)^2 \frac{d^2\eta_0}{d\zeta_0^2} + 2 \zeta_2 \frac{d\eta_0}{d\zeta_0} \right] + \dots \quad (31)$$

Comparing this series for η with equation (17), one finds

$$\eta_1 = \zeta_1 \frac{d\eta_0}{d\zeta_0}, \quad (32)$$

$$2\eta_2 = (\zeta_1)^2 \frac{d^2\eta_0}{d\zeta_0^2} + 2\zeta_2 \frac{d\eta_0}{d\zeta_0}. \quad (33)$$

Expressing $\eta_0 = \eta_0(\xi)$ and $\zeta_0 = \zeta_0(\xi)$ in terms of the parameter ξ , equations (32) and (33) may be re-written as

$$\eta_1 = \zeta_1 \left(\frac{\frac{d\eta_0}{d\xi}}{\frac{d\zeta_0}{d\xi}} \right) \quad (34)$$

and

$$2\eta_2 = 2\zeta_2 \left(\frac{\frac{d\eta_0}{d\xi}}{\frac{d\zeta_0}{d\xi}} \right) + (\zeta_1)^2 \left[\frac{\frac{d^2\eta_0}{d\xi^2} \frac{d\zeta_0}{d\xi} - \frac{d\eta_0}{d\xi} \frac{d^2\zeta_0}{d\xi^2}}{\left(\frac{d\zeta_0}{d\xi} \right)^3} \right], \quad (35)$$

respectively.

A point has now been reached at which a separation of variables may be performed. Any set of complete, orthogonal functions (of μ and/or ξ) would suffice as a set of functions in which to expand the series for χ'_1 , ζ_1 , η_1 , χ'_2 , ζ_2 , and η_2 . However, for this problem the condition that χ'_1 and χ'_2 satisfy Poisson's equation (eqs. [20] and [21]) makes it most convenient to choose, as the particular set of functions to use, the set $[P_j(\mu)]$ of Legendre polynomials. The Legendre polynomial $P_j(\mu)$ of index j is a solution of the differential equation

$$\frac{d}{d\mu} \left((1 - \mu^2) \frac{dP_j}{d\mu} \right) + j(j+1)P_j = 0 \quad (j = 0, 1, 2, \dots) \quad (36)$$

(where by definition $P_0 = 1$). Let it be assumed, then, that χ'_1 , ζ_1 , η_1 , χ'_2 , ζ_2 , and η_2 can be represented by the respective series

$$\chi'_k(\xi, \mu) = \chi'_{k,0}(\xi) + \sum_{j=1}^{\infty} \chi'_{k,j}(\xi) P_j(\mu), \quad (37)$$

$$\zeta_k(\xi, \mu) = \zeta_{k,0}(\xi) + \sum_{j=1}^{\infty} \zeta_{k,j}(\xi) P_j(\mu), \quad (38)$$

$$\eta_k(\xi, \mu) = \eta_{k,0}(\xi) + \sum_{j=1}^{\infty} \eta_{k,j}(\xi) P_j(\mu) \quad (39)$$

Introducing the series (37), (38), and (39) into equations (20), (23), and (25) and making use of (36) and the fact that

$$(1 - \mu^2) = \frac{2}{3} (1 - P_2(\mu)), \quad \frac{dP_2}{d\mu} = 3\mu, \quad (40)$$

one finds by comparing the coefficients of like P_j 's that, to the first order in ϵ ,

$$\frac{1}{\xi^2} \left\{ \frac{d}{d\xi} \left(\xi^2 \frac{d\chi'_{1,j}}{d\xi} \right) - j(j+1) \chi'_{1,j} \right\} = -\zeta_{1,j} \quad (j = 0, 1, 2, \dots), \quad (41)$$

$$\frac{1}{\zeta_0} \left\{ \frac{d\eta_{1,0}}{d\xi} - \frac{\zeta_{1,0}}{\zeta_0} \frac{d\eta_0}{d\xi} \right\} = \frac{d\chi'_{1,0}}{d\xi} + \frac{2}{3} \xi, \quad (42)$$

$$\chi'_{1,j} = \frac{\eta_{1,j}}{\zeta_0} \quad (j \neq 0, 2), \quad (43)$$

$$\chi'_{1,2} = \frac{\eta_{1,2}}{\zeta_0} + \frac{1}{3} \xi^2. \quad (44)$$

One should recall now that the $\chi'_{1,j}$'s are subject to certain boundary conditons, namely, that the gravitational potential functions $\mathfrak{B}'$ for regions inside the star (where $\rho > 0$) shall join smoothly the potential function $\mathfrak{U}'$ for regions outside the star (where $\rho = 0$), i.e., that the functions $\mathfrak{B}'$ and $\mathfrak{U}'$ and their first derivatives shall be equal at the boundary $\xi = \Xi$ of the rotating star. Now, it can be shown that $\mathfrak{U}'$ is expressible in the form

$$\begin{aligned} \mathfrak{U}'(\xi, \mu, \epsilon) &= \mathfrak{U}'_0 + \epsilon \mathfrak{U}'_1 + \epsilon^2 \mathfrak{U}'_2 + \dots \\ &= (4\pi G \rho_c a^2) \left\{ \frac{C_0}{\xi} + \epsilon \left[\sum_{j=0}^{\infty} \frac{C_{1,j}}{\xi^{j+1}} P_j(\mu) \right] \right. \\ &\quad \left. + \epsilon^2 \left[\sum_{j=0}^{\infty} \frac{C_{2,j}}{\xi^{j+1}} P_j(\mu) \right] + \dots \right\} \quad (\xi \geq \Xi). \end{aligned} \quad (45)$$

It can also be shown that the necessary and sufficient conditions that $\mathfrak{U}'$ and $\mathfrak{B}'$ and their first derivatives be equal are (to the first order in ϵ) that the equations

$$\mathfrak{B}'_0(R_0) = \mathfrak{U}'_0(R_0), \quad \left(\frac{d\mathfrak{B}'_0}{dr} \right)_{R_0} = \left(\frac{d\mathfrak{U}'_0}{dr} \right)_{R_0} \quad (46)$$

and

$$\mathfrak{B}'_1(R_0) = \mathfrak{U}'_1(R_0), \quad \left(\frac{d\mathfrak{B}'_1}{dr} \right)_{R_0} = \left(\frac{d\mathfrak{U}'_1}{dr} \right)_{R_0} \quad (47)$$

be satisfied, where the boundary radius $r = R$ is given by

$$R = R_0 + \epsilon R_1(r, \mu) + \epsilon^2 R_2(r, \mu) + \dots \quad (48)$$

In the dimensionless variables these conditions imply that

$$\chi'_0(\xi_0) = \frac{C_0}{\xi_0}, \quad \left(\frac{d\chi'_0}{d\xi} \right)_{\xi_0} = -\frac{C_0}{\xi_0^2}, \quad (49)$$

$$\chi'_1(\xi_0, \mu) = \sum_{j=0}^{\infty} \frac{C_{1,j}}{\xi_0^{j+1}} P_j(\mu), \quad \left(\frac{d\chi'_1}{d\xi} \right)_{\xi_0} = -\sum_{j=0}^{\infty} (j+1) \frac{C_{1,j}}{\xi_0^{j+2}} P_j(\mu), \quad (50)$$

where $\xi = \xi_0$ is the boundary which the star would have if it were not rotating. Combining equations (43) and (44) with (37) and using (50) gives the relations

$$\left(\frac{\eta_{1,j}}{\xi_0} \right)_{\xi_0} - C_{1,j} \frac{1}{\xi_0^{j+1}} = 0, \quad \left[\frac{d}{d\xi} \left(\frac{\eta_{1,j}}{\xi_0} \right) \right]_{\xi_0} + C_{1,j} \frac{(j+1)}{\xi_0^{j+2}} = 0 \quad (j \neq 0, 2) \quad (51)$$

$$\left\{ \begin{aligned} \left(\frac{\eta_{1,2}}{\xi_0} \right)_{\xi_0} - C_{1,2} \frac{1}{\xi_0^3} &= -\frac{1}{3} \xi_0^2, \\ \left[\frac{d}{d\xi} \left(\frac{\eta_{1,2}}{\xi_0} \right) \right]_{\xi_0} + C_{1,2} \frac{3}{\xi_0^4} &= -\frac{2}{3} \xi_0. \end{aligned} \right\} \quad (52)$$

By putting⁵

$$\left(\frac{\eta_{1,j}}{\xi_0} \right) = a_{1,j} \left(\frac{\eta_{1,j}^*}{\xi_0} \right) \quad (j = 1, 2, 3, \dots) \quad (53)$$

⁵ This transformation is justified by the fact that the differential equation for $\eta_{1,2}$ is homogeneous (see eqs. [63]).

and solving the system of equations (51) for $a_{1,j}$, one finds, since the determinant of coefficients does not in general vanish, that

$$a_{1,j} \equiv 0 \quad (j \neq 0, 2), \quad (54)$$

whence by (53) it follows that

$$\eta_{1,j} \equiv 0 \quad (j \neq 0, 2). \quad (55)$$

For $j = 2$, however, (52) gives

$$a_{1,2} = -\frac{5}{3} \left[\frac{\xi^2}{3 \left(\frac{\eta_{1,2}^*}{\xi_0} \right) + \xi \left[\frac{d}{d\xi} \left(\frac{\eta_{1,2}^*}{\xi_0} \right) \right]} \right]_{\xi_0}. \quad (56)$$

Strictly, since $\xi_0 \rightarrow 0$ as $\xi \rightarrow \xi_0$, this should be written

$$a_{1,2} = -\frac{5}{3} \frac{\xi_0^2}{\lim_{\xi \rightarrow \xi_0} \left\{ 3 \left(\frac{\eta_{1,2}^*}{\xi_0} \right) + \xi \left[\frac{d}{d\xi} \left(\frac{\eta_{1,2}^*}{\xi_0} \right) \right] \right\}}, \quad (57)$$

else the expression would contain an indeterminate fraction.

From equation (55) it is evident that

$$\eta_1 = \eta_{1,0} + \eta_{1,2} P_2; \quad (58)$$

and hence by (34) it must be true that

$$\xi_{1,0} = \eta_{1,0} \left(\frac{\frac{d\xi_0}{d\xi}}{\frac{d\eta_0}{d\xi}} \right), \quad \xi_{1,2} = \eta_{1,2} \left(\frac{\frac{d\xi_0}{d\xi}}{\frac{d\eta_0}{d\xi}} \right), \quad (59)$$

or

$$\xi_1 = \xi_{1,0} + \xi_{1,2} P_2, \quad (60)$$

By means of (59) equation (42) can be simplified, since

$$\xi_{1,0} \frac{d\eta_0}{d\xi} = \eta_{1,0} \frac{d\xi_0}{d\xi}.$$

Making this substitution,

$$\left. \begin{aligned} \frac{1}{\xi_0} \left\{ \frac{d\eta_{1,0}}{d\xi} - \frac{\xi_{1,0}}{\xi_0} \frac{d\eta_0}{d\xi} \right\} &= \frac{1}{\xi_0} \frac{d\eta_{1,0}}{d\xi} - \frac{\eta_{1,0}}{\xi_0^2} \frac{d\xi_0}{d\xi} \\ &= \frac{d}{d\xi} \left(\frac{\eta_{1,0}}{\xi_0} \right) = \frac{d\chi'_{1,0}}{d\xi} + \frac{2}{3} \xi. \end{aligned} \right\} \quad (61)$$

Using this and (44) to eliminate the $\chi'_{1,i}$'s from (41), it is found that

$$\frac{1}{\xi^2} \left\{ \frac{d}{d\xi} \left(\xi^2 \frac{d}{d\xi} \left[\frac{\eta_{1,0}}{\xi_0} \right] \right) \right\} = 2 - \xi_{1,0} = 2 - \eta_{1,0} \left(\frac{\frac{d\xi_0}{d\xi}}{\frac{d\eta_0}{d\xi}} \right) \quad (62)$$

and

$$\frac{1}{\xi^2} \left\{ \frac{d}{d\xi} \left(\xi^2 \frac{d}{d\xi} \left[\frac{\eta_{1,2}}{\xi_0} \right] \right) - 6 \left[\frac{\eta_{1,2}}{\xi_0} \right] \right\} = -\xi_{1,2} = -\eta_{1,2} \left(\frac{\frac{d\xi_0}{d\xi}}{\frac{d\eta_0}{d\xi}} \right). \quad (63)$$

Now, equations (62) and (63) are second-order ordinary differential equations, and therefore their general solutions contain two constants which are to be fixed by requiring the solutions to satisfy appropriate initial conditions. If, instead of comparing rotating and nonrotating configurations of the same total mass, as originally proposed, one chooses to compare configurations of equal central density⁶ (and pressure), one then has the initial conditions

$$\eta(\xi=0) = \eta_0(\xi=0) = 1, \quad \left(\frac{d\eta}{d\xi}\right)_{\xi=0} = 0, \quad (64)$$

whence it follows that

$$\eta_{1,0}(\xi=0) = 0, \quad \left(\frac{d\eta_{1,0}}{d\xi}\right)_{\xi=0} = 0, \quad (65)$$

$$\eta_{1,2}(\xi=0) = 0, \quad \left(\frac{d\eta_{1,2}}{d\xi}\right)_{\xi=0} = 0. \quad (66)$$

By the conditions (65) the function $\eta_{1,0}$ is uniquely determined. On the other hand, $\eta_{1,2}$ is still indeterminate to the extent of an arbitrary constant factor, since (63) is linear and homogeneous in $\eta_{1,2}$; and, if $\eta_{1,2}^*$ is a solution of (63), so also is $a_{1,2}\eta_{1,2}^*$. Hence, for convenience, one can choose $\eta_{1,2}^*$ to be approximately proportional to ξ^2 near $\xi=0$. Then, requiring $\eta_{1,2}$ to satisfy the outer boundary conditions for the star, the constant factor, $a_{1,2}$, will be fixed by (57), and the solution becomes completely determinate.

Having formally completed the treatment to the first order in ϵ , it will be much easier now to continue the treatment of the second-order theory. By introducing into equations (21), (24), and (26) the series (37), (38), and (39), one can again perform a separation of variables and this time obtain equations for the second-order radial functions. From (21) it follows directly that

$$\frac{1}{\xi^2} \left\{ \frac{d}{d\xi} \left(\xi^2 \frac{d\chi_{2,j}'}{d\xi} \right) - j(j+1) \chi_{2,j}' \right\} = -\zeta_{2,j} \quad (j=0, 1, 2, \dots). \quad (67)$$

From equations (24) and (26) one finds that

$$\left. \begin{aligned} \frac{1}{\xi_0} \left\{ \frac{d\eta_{2,0}'}{d\xi} - \left(\frac{\zeta_{1,0}}{\xi_0} \frac{d\eta_{1,0}}{d\xi} + \frac{1}{5} \frac{\zeta_{1,2}}{\xi_0} \frac{d\eta_{1,2}}{d\xi} \right) \right. \\ \left. - \frac{d\eta_0}{d\xi} \left[\frac{\zeta_{2,0}}{\xi_0} - \left(\frac{\zeta_{1,0}}{\xi_0} \right)^2 - \frac{1}{5} \left(\frac{\zeta_{1,2}}{\xi_0} \right)^2 \right] \right\} = \frac{d\chi_{2,0}'}{d\xi}, \end{aligned} \right\} \quad (68)$$

$$\frac{\eta_{2,j}}{\xi_0} = \chi_{2,j}' \quad (j \neq 0, 2, 4), \quad (69)$$

$$\frac{\eta_{2,2}}{\xi_0} - \left(\frac{\eta_{1,2}}{\xi_0} \frac{\zeta_{1,0}}{\xi_0} + \frac{1}{7} \frac{\eta_{1,2}}{\xi_0} \frac{\zeta_{1,2}}{\xi_0} \right) = \chi_{2,2}', \quad (70)$$

$$\frac{\eta_{2,4}}{\xi_0} - \frac{9}{35} \frac{\eta_{1,2}}{\xi_0} \frac{\zeta_{1,2}}{\xi_0} = \chi_{2,4}'. \quad (71)$$

Equation (68) can be simplified by making use of (33). Expanding the coefficient ξ_1^2 ,

$$\zeta_{2,0} \frac{d\eta_0}{d\xi} = \eta_{2,0} \frac{d\zeta_0}{d\xi} - \frac{1}{2} [(\zeta_{1,0})^2 + \frac{1}{5} (\zeta_{1,2})^2] \frac{d^2\eta_0}{d\xi^2} \frac{d\zeta_0}{d\xi}$$

⁶ Comparison of rotating and nonrotating configurations having the same central density is much simpler than comparison of configurations of equal total mass. Obviously, however, given a solution of either problem, a solution of the other can be obtained from it.

or

$$\zeta_{2,0} \frac{d\eta_0}{d\xi} = \eta_{2,0} \frac{d\zeta_0}{d\xi} - \frac{1}{2} \left\{ \left[\frac{d\eta_{1,0}}{d\xi} \zeta_{1,0} - \eta_{1,0} \frac{d\zeta_{1,0}}{d\xi} \right] + \frac{1}{5} \left[\frac{d\eta_{1,2}}{d\xi} \zeta_{1,2} - \eta_{1,2} \frac{d\zeta_{1,2}}{d\xi} \right] \right\} \quad (72)$$

Using this and (59) in (68) gives

$$\frac{d}{d\xi} \left(\frac{\eta_{2,0}}{\zeta_0} - \frac{1}{2} \frac{\zeta_{1,0}}{\zeta_0} \frac{\eta_{1,0}}{\zeta_0} - \frac{1}{10} \frac{\zeta_{1,2}}{\zeta_0} \frac{\eta_{1,2}}{\zeta_0} \right) = \frac{d}{d\xi} (\chi'_{2,0}) \quad (73)$$

Let it be recalled, once again, that χ' must satisfy certain boundary conditions which, to the second order in ϵ , require that

$$\mathfrak{B}'_2(R_0) = \mathfrak{U}'_2(R_0), \quad \left(\frac{d\mathfrak{B}'_2}{d\mathbf{r}} \right)_{R_0} = \left(\frac{d\mathfrak{U}'_2}{d\mathbf{r}} \right)_{R_0}, \quad (74)$$

where, as before, R_0 is the boundary of the corresponding stationary configuration. Since, by equation (45), $\mathfrak{U}'_2$ has the form

$$\mathfrak{U}'_2 = (4\pi G \rho_c a^2) \sum_{j=0}^{\infty} \frac{C_{2,j}}{\xi_0^{j+1}} P_j(\mu),$$

one has from equations (69), (70), (71), and (73) and the conditions (74),

$$\left. \begin{aligned} \left(\frac{\eta_{2,j}}{\zeta_0} \right)_{\xi_0} - C_{2,j} \frac{1}{\xi_0^{j+1}} &= 0, \\ \left[\frac{d}{d\xi} \left(\frac{\eta_{2,j}}{\zeta_0} \right) \right]_{\xi_0} + C_{2,j} \frac{j+1}{\xi_0^{j+2}} &= 0 \end{aligned} \right\} \quad (j \neq 0, 2, 4), \quad (75)$$

$$\left. \begin{aligned} \left(\frac{\eta_{2,0}}{\zeta_0} \right)_{\xi_0} - C_{2,0} \frac{1}{\xi_0} &= \frac{1}{2} \left(\frac{\zeta_{1,0}}{\zeta_0} \frac{\eta_{1,0}}{\zeta_0} + \frac{1}{5} \frac{\zeta_{1,2}}{\zeta_0} \frac{\eta_{1,2}}{\zeta_0} \right)_{\xi_0} = T_{2,0}(\xi_0), \\ \left[\frac{d}{d\xi} \left(\frac{\eta_{2,0}}{\zeta_0} \right) \right]_{\xi_0} + C_{2,0} \frac{1}{\xi_0^2} &= \left(\frac{dT_{2,0}}{d\xi} \right)_{\xi_0}, \end{aligned} \right\} \quad (76)$$

$$\left. \begin{aligned} \left(\frac{\eta_{2,2}}{\zeta_0} \right)_{\xi_0} - C_{2,2} \frac{1}{\xi_0^3} &= \left(\frac{\zeta_{1,0}}{\zeta_0} \frac{\eta_{1,2}}{\zeta_0} + \frac{1}{7} \frac{\zeta_{1,2}}{\zeta_0} \frac{\eta_{1,2}}{\zeta_0} \right)_{\xi_0} = T_{2,2}(\xi_0), \\ \left[\frac{d}{d\xi} \left(\frac{\eta_{2,2}}{\zeta_0} \right) \right]_{\xi_0} + C_{2,2} \frac{3}{\xi_0^4} &= \left(\frac{dT_{2,2}}{d\xi} \right)_{\xi_0}, \end{aligned} \right\} \quad (77)$$

$$\left. \begin{aligned} \left(\frac{\eta_{2,4}}{\zeta_0} \right)_{\xi_0} - C_{2,4} \frac{1}{\xi_0^5} &= \frac{9}{35} \left(\frac{\zeta_{1,2}}{\zeta_0} \frac{\eta_{1,2}}{\zeta_0} \right)_{\xi_0} = T_{2,4}(\xi_0), \\ \left[\frac{d}{d\xi} \left(\frac{\eta_{2,4}}{\zeta_0} \right) \right]_{\xi_0} + C_{2,4} \frac{5}{\xi_0^6} &= \left(\frac{dT_{2,4}}{d\xi} \right)_{\xi_0}. \end{aligned} \right\} \quad (78)$$

Setting

$$\eta_{2,j} = a_{2,j} \cdot \eta_{2,j}^*, \quad (79)$$

and solving (75) for $a_{2,j}$ gives

$$a_{2,j} \equiv 0 \quad (j \neq 0, 2, 4), \quad (80)$$

whence it follows that

$$\eta_{2,j} \equiv 0 \quad (j \neq 0, 2, 4). \quad (81)$$

This, with (35), shows that

$$\zeta_{2,j} \equiv 0 \quad (j \neq 0, 2, 4). \quad (82)$$

Further, by solving (76), (77), and (78) one finds that

$$a_{2,0} = \left[\frac{T_{2,0} + \xi \frac{dT_{2,0}}{d\xi}}{\frac{\eta_{2,0}^*}{\zeta_0} + \xi \frac{d}{d\xi} \left(\frac{\eta_{2,0}^*}{\zeta_0} \right)} \right]_{\xi_0}, \quad (83)$$

$$a_{2,2} = \left[\frac{3T_{2,2} + \xi \frac{dT_{2,2}}{d\xi}}{3 \frac{\eta_{2,2}^*}{\zeta_0} + \xi \frac{d}{d\xi} \left(\frac{\eta_{2,2}^*}{\zeta_0} \right)} \right]_{\xi_0}, \quad (84)$$

$$a_{2,4} = \left[\frac{5T_{2,4} + \xi \frac{dT_{2,4}}{d\xi}}{5 \frac{\eta_{2,4}^*}{\zeta_0} + \xi \frac{d}{d\xi} \left(\frac{\eta_{2,4}^*}{\zeta_0} \right)} \right]_{\xi_0}. \quad (85)$$

Now, by means of (70), (71), and (73) one can eliminate the $\chi'_{2,i}$'s from (67); and, simultaneously eliminating $\zeta_{2,i}$ ($j = 0, 2, 4$) by means of (35), one has the final equations

$$\left. \begin{aligned} \frac{1}{\xi^2} \left\{ \frac{d}{d\xi} \left(\xi^2 \frac{d}{d\xi} \left[\frac{\eta_{2,0}}{\zeta_0} \right] \right) \right\} &= - \left\{ \eta_{2,0} \left(\frac{\frac{d\zeta_0}{d\xi}}{\frac{d\eta_0}{d\xi}} \right) \right. \\ &\quad - \frac{1}{2 \frac{d\eta_0}{d\xi}} \left[\left(\frac{d\eta_{1,0}}{d\xi} \zeta_{1,0} - \eta_{1,0} \frac{d\zeta_{1,0}}{d\xi} \right) + \frac{1}{5} \left(\frac{d\eta_{1,2}}{d\xi} \zeta_{1,2} - \eta_{1,2} \frac{d\zeta_{1,2}}{d\xi} \right) \right] \\ &\quad \left. + \frac{1}{2 \xi^2} \left\{ \frac{d}{d\xi} \left(\xi^2 \frac{d}{d\xi} \left[\frac{\zeta_{1,0}}{\zeta_0} \frac{\eta_{1,0}}{\zeta_0} + \frac{1}{5} \frac{\zeta_{1,2}}{\zeta_0} \frac{\eta_{1,2}}{\zeta_0} \right] \right) \right\} \right\}, \end{aligned} \right\} \quad (86)$$

$$\left. \begin{aligned} \frac{1}{\xi^2} \left\{ \frac{d}{d\xi} \left(\xi^2 \frac{d}{d\xi} \left[\frac{\eta_{2,2}}{\zeta_0} \right] \right) - 6 \left[\frac{\eta_{2,2}}{\zeta_0} \right] \right\} &= - \left\{ \eta_{2,2} \left(\frac{\frac{d\zeta_0}{d\xi}}{\frac{d\eta_0}{d\xi}} \right) \right. \\ &\quad - \frac{1}{\frac{d\eta_0}{d\xi}} \left[\left(\zeta_{1,2} \frac{d\eta_{1,0}}{d\xi} - \eta_{1,2} \frac{d\zeta_{1,0}}{d\xi} \right) + \frac{1}{7} \left(\zeta_{1,2} \frac{d\eta_{1,2}}{d\xi} - \eta_{1,2} \frac{d\zeta_{1,2}}{d\xi} \right) \right] \\ &\quad \left. + \frac{1}{\xi_0} \left\{ \frac{d}{d\xi} \left(\xi^2 \frac{d}{d\xi} \left[\frac{\eta_{1,2}}{\zeta_0} \frac{\zeta_{1,0}}{\zeta_0} + \frac{1}{7} \frac{\eta_{1,2}}{\zeta_0} \frac{\zeta_{1,2}}{\zeta_0} \right] \right) - 6 \left[\frac{\eta_{1,2}}{\zeta_0} \frac{\zeta_{1,0}}{\zeta_0} + \frac{1}{7} \frac{\eta_{1,2}}{\zeta_0} \frac{\zeta_{1,2}}{\zeta_0} \right] \right\} \right\}, \end{aligned} \right\} \quad (87)$$

$$\begin{aligned}
 & \frac{1}{\xi^2} \left\{ \frac{d}{d\xi} \left(\xi^2 \frac{d}{d\xi} \left[\frac{\eta_{2,4}}{\xi_0} \right] \right) - 20 \left[\frac{\eta_{2,4}}{\xi_0} \right] \right\} \\
 &= - \left\{ \eta_{2,4} \left(\frac{\frac{d\xi_0}{d\xi}}{\frac{d\eta_0}{d\xi}} \right) - \frac{1}{\frac{d\eta_0}{d\xi}} \left[\frac{9}{35} \left(\frac{d\eta_{1,2}}{d\xi} \xi_{1,2} - \eta_{1,2} \frac{d\xi_{1,2}}{d\xi} \right) \right] \right\} \\
 &+ \frac{1}{\xi^2} \left\{ \frac{d}{d\xi} \left(\xi^2 \frac{d}{d\xi} \left[\frac{9}{35} \frac{\eta_{1,2}}{\xi_0} \frac{\xi_{1,2}}{\xi_0} \right] \right) - 20 \left[\frac{9}{35} \frac{\eta_{1,2}}{\xi_0} \frac{\xi_{1,2}}{\xi_0} \right] \right\}. \quad (88)
 \end{aligned}$$

As in the case of the first-order theory, the solutions for $\eta_{2,j}$ must satisfy certain initial conditions, namely,

$$\eta_{2,j}(\xi=0) = 0, \quad \left(\frac{d\eta_{2,j}}{d\xi} \right)_{\xi_0} = 0 \quad (j=0, 2, 4).$$

Besides these, the boundary conditions (83), (84), and (85) must be fulfilled.

The boundary of the configuration will be defined by the condition

$$P(R) = P_c \eta(\Xi) = 0 \quad (89)$$

or

$$\eta_0(\Xi) + \epsilon [\eta_{1,0} + \eta_{1,2} P_2]_{\Xi} + \epsilon^2 [\eta_{2,0} + \eta_{2,2} P_2 + \eta_{2,4} P_4]_{\Xi} + \dots = 0.$$

But, since $\eta_0(\xi_0) = 0$,

$$\eta_0(\Xi) = (\Xi - \xi_0) \left(\frac{d\eta_0}{d\xi} \right)_{\xi_0} + \frac{1}{2} (\Xi - \xi_0)^2 \left(\frac{d^2\eta_0}{d\xi^2} \right)_{\xi_0} + \dots$$

Similarly,

$$\eta_{1,j}(\Xi) = \eta_{1,j}(\xi_0) + (\Xi - \xi_0) \left(\frac{d\eta_{1,j}}{d\xi} \right)_{\xi_0} + \dots \quad (j=0, 2)$$

and

$$\eta_{2,j}(\Xi) = \eta_{2,j}(\xi_0) + \dots \quad (j=0, 2, 4).$$

Therefore, to the second order in ϵ (since $[\Xi - \xi_0] \propto \epsilon$),

$$\begin{aligned}
 & (\Xi - \xi_0) \left(\frac{d\eta_0}{d\xi} \right)_{\xi_0} + \frac{1}{2} (\Xi - \xi_0)^2 \left(\frac{d^2\eta_0}{d\xi^2} \right)_{\xi_0} + \epsilon [\eta_{1,0} + \eta_{1,2} P_2]_{\xi_0} \\
 & + \epsilon (\Xi - \xi_0) \left[\frac{d\eta_{1,0}}{d\xi} + \frac{d\eta_{1,2}}{d\xi} P_2 \right]_{\xi_0} \\
 & + \epsilon^2 [\eta_{2,0} + \eta_{2,2} P_2 + \eta_{2,4} P_4]_{\xi_0} + \dots = 0.
 \end{aligned}$$

But, to the first order in ϵ (i.e., taking only terms to the first power in ϵ or $[\Xi - \xi_0]$), it is evident that

$$\Xi - \xi_0 = - \frac{\epsilon}{\left(\frac{d\eta_0}{d\xi} \right)_{\xi_0}} [\eta_{1,0} + \eta_{1,2} P_2]_{\xi_0}. \quad (90)$$

Therefore, setting

$$\frac{1}{2} (\Xi - \xi_0)^2 = - \frac{1}{2} \epsilon \frac{(\Xi - \xi_0)}{\left(\frac{d\eta_0}{d\xi} \right)_{\xi_0}} [\eta_{1,0} + \eta_{1,2} P_2]_{\xi_0},$$

one has, to the second order in ϵ ,

$$\begin{aligned} (\Xi - \xi_0) \left\{ \frac{d\eta_0}{d\xi} - \frac{1}{2} \epsilon \frac{1}{\frac{d\eta_0}{d\xi}} [\eta_{1,0} + \eta_{1,2} P_2] \frac{d^2\eta_0}{d\xi^2} \right. \\ \left. + \epsilon \left[\frac{d\eta_{1,0}}{d\xi} + \frac{d\eta_{1,2}}{d\xi} P_2 \right] \right\}_{\xi_0} + \epsilon [\eta_{1,0} + \eta_{1,2} P_2]_{\xi_0} \\ + \epsilon^2 [\eta_{2,0} + \eta_{2,2} P_2 + \eta_{2,4} P_4]_{\xi_0} + \dots = 0 \end{aligned}$$

or

where

$$\Xi = \xi_0 + \epsilon \xi_1 + \epsilon^2 \xi_2 + \dots,$$

$$\xi_1 = - \left\{ \frac{1}{\frac{d\eta_0}{d\xi}} [\eta_{1,0} + \eta_{1,2} P_2] \right\}_{\xi_0}, \quad (91)$$

$$\begin{aligned} \xi_2 = - \frac{1}{\left(\frac{d\eta_0}{d\xi} \right)_{\xi_0}} \left\{ [\eta_{2,0} + \eta_{2,2} P_2 + \eta_{2,4} P_4] \right. \\ \left. + \frac{1}{\frac{d\eta_0}{d\xi}} \left[\frac{1}{2} \frac{\frac{d^2\eta_0}{d\xi^2}}{\frac{d\eta_0}{d\xi}} - \left(\frac{d\eta_{1,0}}{d\xi} + \frac{d\eta_{1,2}}{d\xi} P_2 \right) [\eta_{1,0} + \eta_{1,2} P_2] \right] \right\}_{\xi_0}. \end{aligned} \quad (92)$$

The mass of a star whose density distribution is $\rho = \rho(r, \mu)$ will be given by the integral

$$M = 2\pi \int_0^R \int_{-1}^1 \rho(r, \mu) r^2 dr d\mu.$$

Since

$$\int_{-1}^1 P_j(\mu) d\mu = 0.$$

the mass of a rotating star will (to the second order in ϵ) be

$$\begin{aligned} M &= 2\pi a^3 \rho_c \int_0^\Xi \int_{-1}^1 \xi \xi^2 d\xi d\mu \\ &= 4\pi a^3 \rho_c \left\{ \int_0^{\xi_0} (\xi_0 + \epsilon \xi_{1,0} + \epsilon^2 \xi_{2,0}) \xi^2 d\xi + \int_{\xi_0}^\Xi (\xi_0 + \epsilon \xi_{1,0} + \epsilon^2 \xi_{2,0}) \xi^2 d\xi \right\}. \end{aligned}$$

Letting

$$\zeta_0 = \xi_0(\xi_0) + (\xi - \xi_0) \left(\frac{d\xi_0}{d\xi} \right)_{\xi_0} + \frac{1}{2} (\xi - \xi_0)^2 \left(\frac{d^2\xi_0}{d\xi^2} \right)_{\xi_0} + \dots,$$

and similarly for $\zeta_{1,0}$ and $\zeta_{2,0}$, one has (to the second order in ϵ) simply

$$M = 4\pi a^3 \rho_c \left\{ \int_0^{\xi_0} (\xi_0 + \epsilon \xi_{1,0} + \epsilon^2 \xi_{2,0}) \xi^2 d\xi \right\}, \quad (93)$$

since the only part of the second term not higher than the second order in ϵ vanishes. Now, introducing into (93) the expressions for $\xi_{1,0}$ and $\xi_{2,0}$ found from equations (62), (67), and (73), and that for ξ_0 found from (19) and (22), one has for the mass

$$M = -4\pi\alpha^3\rho_c \left\{ \begin{aligned} & \left[\frac{\xi^2}{\xi_0} \frac{d\eta_0}{d\xi} + \epsilon \left(-\frac{2}{3}\xi^3 + \xi^2 \frac{d}{d\xi} \left[\frac{\eta_{1,0}}{\xi_0} \right] \right) \right. \\ & \left. + \epsilon^2 \left(\xi^2 \frac{d}{d\xi} \left[\frac{\eta_{2,0}}{\xi_0} - \frac{1}{2} \frac{\xi_{1,0}}{\xi_0} \frac{\eta_{1,0}}{\xi_0} - \frac{1}{10} \frac{\xi_{1,2}}{\xi_0} \frac{\eta_{1,2}}{\xi_0} \right] \right) + \dots \right]_{\xi_0} \end{aligned} \right\} \quad (94)$$

or, in terms of the mass M_0 of the static configuration,

$$M = M_0 + 4\pi\alpha^3\rho_c \left\{ \begin{aligned} & \left[-\epsilon \left(\xi^2 \frac{d}{d\xi} \left[\frac{\eta_{1,0}}{\xi_0} \right] - \frac{2}{3}\xi^3 \right) \right. \\ & \left. - \epsilon^2 \left(\xi^2 \frac{d}{d\xi} \left[\frac{\eta_{2,0}}{\xi_0} - \frac{1}{2} \frac{\xi_{1,0}}{\xi_0} \frac{\eta_{1,0}}{\xi_0} - \frac{1}{10} \frac{\xi_{1,2}}{\xi_0} \frac{\eta_{1,2}}{\xi_0} \right] \right) + \dots \right]_{\xi_0} \end{aligned} \right\} \quad (95)$$

The total volume of the configuration will be

$$\begin{aligned} V &= 2\pi \int_0^R \int_{-1}^1 r^2 dr d\mu = 4\pi\alpha^3 \int_0^\Xi \xi^2 d\xi \\ &= \frac{4}{3}\pi\alpha^3 [\xi_0 + \epsilon\xi'_1 + \epsilon^2\xi'_2 + \dots]^3 \end{aligned}$$

or

$$V = \frac{4}{3}\pi\alpha^3 [\xi_0^3 + 3\epsilon\xi_0^2\xi'_1 + \epsilon^2(2\xi_0^2\xi'_2 + 3\xi_0\xi_1'^2) + \dots], \quad (96)$$

where

$$\xi'_1 = - \left(\frac{\eta_{1,0}}{\frac{d\eta_0}{d\xi}} \right)_{\xi_0}, \quad (97)$$

$$\xi'_2 = - \frac{1}{\left(\frac{d\eta_0}{d\xi} \right)_{\xi_0}} \left\{ \eta_{2,0} + \frac{\eta_{1,0}}{\frac{d\eta_0}{d\xi}} \left[\frac{1}{2} \left(\frac{\frac{d^2\eta_0}{d\xi^2}}{\frac{d\eta_0}{d\xi}} \right) - \frac{d\eta_{1,0}}{d\xi} \right] \right\}_{\xi_0}. \quad (98)$$

In terms of the volume V_0 of the unperturbed configuration this is

$$V = V_0 + \epsilon[4\pi\alpha^3\xi_0^2\xi'_1] + \epsilon^2[4\pi\alpha^3(\frac{2}{3}\xi_0^2\xi'_2 + \xi_0\xi_1'^2)] + \dots \quad (99)$$

The mean density will be

$$\bar{\rho} = \frac{M}{V} = \frac{M_0}{V_0} \left\{ 1 + \epsilon\bar{\rho}_1 + \epsilon^2\bar{\rho}_2 + \dots \right\}, \quad (100)$$

where

$$\bar{\rho}_1 = \frac{M_1}{M_0} - \frac{V_1}{V_0}, \quad \bar{\rho}_2 = \left[\frac{M_2}{M_0} - \frac{V_2}{V_0} \right] - \frac{V_1}{V_0} \left[\frac{M_1}{M_0} - \frac{V_1}{V_0} \right], \quad (101)$$

$$\left. \begin{aligned} M_1 &= -4\pi\alpha^3\rho_c \left(\xi^2 \frac{d}{d\xi} \left[\frac{\eta_{1,0}}{\xi_0} \right] - \frac{2}{3}\xi^3 \right)_{\xi_0}, \\ M_2 &= -4\pi\alpha^3\rho_c \left(\xi^2 \frac{d}{d\xi} \left[\frac{\eta_{2,0}}{\xi_0} - \frac{1}{2} \frac{\xi_{1,0}}{\xi_0} \frac{\eta_{1,0}}{\xi_0} - \frac{1}{10} \frac{\xi_{1,2}}{\xi_0} \frac{\eta_{1,2}}{\xi_0} \right] \right)_{\xi_0}, \\ V_1 &= 4\pi\alpha^3\xi_0^2\xi'_1, \quad V_2 = 4\pi\alpha^3(\frac{2}{3}\xi_0^2\xi'_2 + \xi_0\xi_1'^2). \end{aligned} \right\} \quad (102)$$

Consider two stars (on the same model) having the same total mass M . Let one be rotating with angular velocity ω and let the other be stationary. Then

$$M' = M = M''$$

or

$$-4\pi a^3 \rho'_c \left(\frac{\xi^2}{\xi_0} \frac{d\eta_0}{d\xi} \right)_{\xi_0} = -4\pi a^3 \rho''_c \left[\frac{\xi^2}{\xi_0} \frac{d\eta_0}{d\xi} + \epsilon \left(-\frac{2}{3} \xi^3 + \xi^2 \frac{d}{d\xi} \left[\frac{\eta_{1,0}}{\xi_0} \right] \right) \right. \\ \left. + \epsilon^2 \left(\xi^2 \frac{d}{d\xi} \left[\frac{\eta_{2,0}}{\xi_0} - \frac{1}{2} \frac{\xi_{1,0}}{\xi_0} \frac{\eta_{1,0}}{\xi_0} - \frac{1}{10} \frac{\xi_{1,2}}{\xi_0} \frac{\eta_{1,2}}{\xi_0} \right] \right) \right]_{\xi_0},$$

where ρ'_c is the central density of the nonrotating configuration and ρ''_c the central density of the rotating configuration. The relation between them can be more conveniently written as

$$\frac{\rho'_c}{\rho''_c} = 1 + \left\{ \frac{\xi_0}{\frac{d\eta_0}{d\xi}} \left\{ \epsilon \left(-\frac{2}{3} \xi + \frac{d}{d\xi} \left[\frac{\eta_{1,0}}{\xi_0} \right] \right) + \epsilon^2 \frac{d}{d\xi} \left[\frac{\eta_{2,0}}{\xi_0} - \frac{1}{2} \frac{\xi_{1,0}}{\xi_0} \frac{\eta_{1,0}}{\xi_0} - \frac{1}{10} \frac{\xi_{1,2}}{\xi_0} \frac{\eta_{1,2}}{\xi_0} \right] \right\} \right\}_{\xi_0} \quad (103)$$

Since $d\eta_0/d\xi$ and $d/d\xi [\eta_{1,0}/\xi_0]$ are both negative at $\xi = \xi_0$, it is clear that the central density of the nonrotating configuration must be the greater (at least for small angular velocities).

II. A CONFIGURATION SUBJECT TO THE TIDAL ACTION OF AN INVERSE-SQUARE FIELD

Let a configuration be placed in the inverse-square force-field of a point-mass which is at a distance R from the center of mass of the configuration. The effect of the superposed force-field will be twofold. First, the configuration as a whole will be accelerated to move approximately as a particle of the same mass situated at the configuration's center of mass. Second, because of the configuration's extension, the differential effects caused by the nonuniformity of the field will result in tidal effects in the configuration. It is these latter effects which are of especial interest.

The general equations of the problem are the same as before, where now

$$\mathfrak{B} = \mathfrak{B}' + \mathfrak{B}'' \quad (104)$$

and $\mathfrak{B}''$ is the tidal potential function. Evidently one must find the explicit expression for the tidal potential function $\mathfrak{B}''$. Now the general potential function for the inverse-square field of a particle of mass M'' which is at a distance R in the direction corresponding to $\mu = 1$ is, taking the origin of co-ordinates at the center of mass of the configuration,

$$\mathfrak{B}'' = \frac{GM''}{R} \left[1 - 2\mu \left(\frac{r}{R} \right) + \left(\frac{r}{R} \right)^2 \right]^{1/2}, \quad (105)$$

or, expanding the right member in Legendre polynomials P_j ,

$$\mathfrak{B}'' = \frac{GM''}{R} \left\{ 1 + \frac{r}{R} \mu + \sum_{j=2}^{\infty} \left(\frac{r}{R} \right)^j P_j(\mu) \right\}. \quad (106)$$

As is well known, the problem can be reduced to one in statics by introducing "inertial" accelerations equal and opposite to the "field" accelerations; this corresponds to the addition of a potential

$$\mathfrak{B}^* = -\frac{GM''}{R^2} r \mu \quad (107)$$

to the potential $\mathfrak{B}^{**}$. The result is the tidal potential

$$\bar{\mathfrak{B}}'' = \frac{GM''}{R} \left[1 + \sum_{j=2}^{\infty} \left(\frac{r}{R} \right)^j P_j(\mu) \right].$$

This is the tidal potential of a point-mass. Of more interest is the tidal potential of a second star. This, inclusive of only first-order effects,⁷ is given by

$$\mathfrak{B}'' = \frac{GM''}{R} \left[1 + \sum_{j=2}^4 \left(\frac{r}{R} \right)^j P_j(\mu) \right]. \quad (108)$$

Let

$$\mathfrak{B}' = (4\pi G \rho_c a^2) \chi', \quad \mathfrak{B}'' = (4\pi G \rho_c a^2) \chi'', \quad R = a \tau. \quad (109)$$

Then

$$\chi'' = \chi_0'' \left[1 + \sum_{j=2}^4 \left(\frac{\xi}{\tau} \right)^j P_j(\mu) \right], \quad (110)$$

where

$$\chi_0'' = \frac{M''}{4\pi \rho_c a^3 \tau}. \quad (111)$$

Now the potential $\mathfrak{B}^{**}$ must satisfy Poisson's equation (eq. [3]). The function $\mathfrak{B}^{**}$ given in (106) satisfies this equation identically; so also does $\mathfrak{B}''$, given in (108).

Equations (1), (3), (6), and (110) are all the necessary equations for this problem. In dimensionless variables they become

$$\eta = \eta(\xi), \quad (112)$$

$$\frac{1}{\xi} \frac{\partial \eta}{\partial \xi} = \frac{\partial \chi'}{\partial \xi} + \frac{1}{\tau} \chi_0'' \sum_{j=2}^4 j \left(\frac{\xi}{\tau} \right)^{j-1} P_j(\mu), \quad (113)$$

$$\frac{1}{\xi} \frac{\partial \eta}{\partial \mu} = \frac{\partial \chi'}{\partial \mu} + \chi_0'' \sum_{j=2}^4 \frac{dP_j}{d\mu} \left(\frac{\xi}{\tau} \right)^j, \quad (114)$$

$$\frac{1}{\xi^2} \left\{ \frac{\partial}{\partial \xi} \left(\xi^2 \frac{\partial \chi'}{\partial \xi} \right) + \frac{\partial}{\partial \mu} \left((1 - \mu^2) \frac{\partial \chi'}{\partial \mu} \right) \right\} = -\xi, \quad (115)$$

$$\frac{1}{\xi^2} \left\{ \frac{\partial}{\partial \xi} \left(\xi^2 \frac{\partial \chi''}{\partial \xi} \right) + \frac{\partial}{\partial \mu} \left((1 - \mu^2) \frac{\partial \chi''}{\partial \mu} \right) \right\} = 0. \quad (116)$$

If χ_0'' is small enough (i.e., if M'' is small enough or if τ is large enough, or both), then one can expand ξ , η , and χ' in a power series in χ_0'' . Then we can write

$$\xi = \xi_0(\xi) + (\chi_0'') \xi_1(\xi, \mu) + (\chi_0'')^2 \xi_2(\xi, \mu) + \dots, \quad (117)$$

⁷ See Chandrasekhar, *M.N.*, **93**, 450, 1933.

$$\eta = \eta_0(\xi) + (\chi_0'') \eta_1(\xi, \mu) + (\chi_0'')^2 \eta_2(\xi, \mu) + \dots, \quad (118)$$

$$\chi' = \chi_0'(\xi) + (\chi_0'') \chi_1'(\xi, \mu) + (\chi_0'')^2 \chi_2'(\xi, \mu) + \dots. \quad (119)$$

Introducing (117)–(119) into (113)–(116) and retaining only terms of the first order in χ_0'' , one obtains

$$\frac{1}{\xi^2} \left\{ \frac{d}{d\xi} \left(\xi^2 \frac{d\chi_0'}{d\xi} \right) \right\} = -\zeta_0, \quad (120)$$

$$\frac{1}{\xi^2} \left\{ \frac{\partial}{\partial \xi} \left(\xi^2 \frac{\partial \chi_1'}{\partial \xi} \right) + \frac{\partial}{\partial \mu} \left((1 - \mu^2) \frac{\partial \chi_1'}{\partial \mu} \right) \right\} = -\zeta_1, \quad (121)$$

$$\frac{1}{\zeta_0} \frac{d\eta_0}{d\xi} = \frac{d\chi_0'}{d\xi}, \quad (122)$$

$$\frac{1}{\zeta_0} \left\{ \frac{\partial \eta_1}{\partial \xi} - \left(\frac{\zeta_1}{\zeta_0} \right) \frac{d\eta_0}{d\xi} \right\} = \frac{\partial \chi_1'}{d\xi} + \frac{1}{\tau} \sum_{j=2}^4 j \left(\frac{\xi}{\tau} \right)^{j-1} P_j(\mu), \quad (123)$$

$$\frac{1}{\zeta_0} \frac{\partial \eta_1}{\partial \mu} = \frac{\partial \chi_1'}{\partial \mu} + \sum_{j=2}^4 \left(\frac{\xi}{\tau} \right)^j \frac{dP_j}{d\mu}. \quad (124)$$

Separating the variables, as before, by taking

$$\zeta_1 = \sum_{j=0}^{\infty} \zeta_{1,j}(\xi) P_j(\mu), \quad \eta_1 = \sum_{j=0}^{\infty} \eta_{1,j}(\xi) P_j(\mu), \quad \chi_1' = \sum_{j=0}^{\infty} \chi_{1,j}'(\xi) P_j(\mu), \quad (125)$$

one obtains from (121), (123), and (124) the results

$$\frac{1}{\xi^2} \left\{ \frac{d}{d\xi} \left(\xi^2 \frac{d\chi_{1,j}'}{d\xi} \right) - j(j+1) \chi_{1,j}' \right\} = -\zeta_{1,j} \quad (j = 0, 1, 2, \dots), \quad (126)$$

$$\frac{1}{\zeta_0} \left\{ \frac{d\eta_{1,0}}{d\xi} - \left(\frac{\zeta_{1,0}}{\zeta_0} \right) \frac{d\eta_0}{d\xi} \right\} = \frac{d\chi_{1,0}'}{d\xi}, \quad (127)$$

$$\frac{\eta_{1,j}}{\zeta_0} = \chi_{1,j}' \quad (j = 1, 5, 6, \dots), \quad (128)$$

$$\frac{\eta_{1,j}}{\zeta_0} = \chi_{1,j}' + \left(\frac{\xi}{\tau} \right)^j \quad (j = 2, 3, 4). \quad (129)$$

Now the existence of a pressure-density relation such as (112) is known (from [34]) to imply the existence of a relation

$$\eta_{1,j} \frac{d\zeta_0}{d\xi} = \zeta_{1,j} \frac{d\eta_0}{d\xi} \quad (j = 0, 1, 2, \dots). \quad (130)$$

Substituting this in (127) gives

$$\frac{1}{\zeta_0} \left\{ \frac{d\eta_{1,0}}{d\xi} - \frac{\zeta_{1,0}}{\zeta_0} \frac{d\eta_0}{d\xi} \right\} = \frac{1}{\zeta_0} \frac{d\eta_{1,0}}{d\xi} - \frac{\eta_{1,0}}{\zeta_0^2} \frac{d\zeta_0}{d\xi} = \frac{d}{d\xi} \left(\frac{\eta_{1,0}}{\zeta_0} \right) = \frac{d}{d\xi} (\chi_{1,0}'). \quad (131)$$

Now the external potential function

$$W' = (4\pi G \rho_c a^2) \left[\frac{B_0}{\xi} + \chi_0'' \sum_{j=0}^{\infty} \frac{B_{1,j}}{\xi^{j+1}} P_j(\mu) + \dots \right] \quad (\xi \geq \Xi) \quad (131)$$

and its first derivative must equal, respectively, the internal potential and its first derivative at the boundary $\xi = \Xi$ of the distorted configuration. Combining equation (125) with (128) and (129), these conditions imply the relations

$$\chi_0'(\xi_0) = \frac{B_0}{\xi_0}, \quad \left(\frac{d\chi_0'}{d\xi} \right)_{\xi_0} = -\frac{B_0}{\xi_0^2}, \quad (132)$$

$$\left. \begin{aligned} \left(\frac{\eta_{1,j}}{\xi_0} \right)_{\xi_0} - \frac{B_{1,j}}{\xi_0^{j+1}} &= \left(\frac{\xi_0}{\tau} \right)^j, \\ \left[\frac{d}{d\xi} \left(\frac{\eta_{1,j}}{\xi_0} \right) \right]_{\xi_0} + (j+1) \frac{B_{1,j}}{\xi_0^{j+1}} &= \frac{j}{\tau} \left(\frac{\xi_0}{\tau} \right)^{j-1} \end{aligned} \right\} \quad (j = 2, 3, 4), \quad (133)$$

and⁸

$$\left. \begin{aligned} \left(\frac{\eta_{1,j}}{\xi_0} \right)_{\xi_0} - \frac{B_{1,j}}{\xi_0^{j+1}} &= 0, \\ \left[\frac{d}{d\xi} \left(\frac{\eta_{1,j}}{\xi_0} \right) \right]_{\xi_0} + \frac{B_{1,j}}{\xi_0^{j+2}} &= 0 \end{aligned} \right\} \quad (j = 0, 1, 5, 6, \dots). \quad (134)$$

Setting⁹

$$\eta_{1,j} = a_{1,j}^*,$$

and solving (133) and (134) for $a_{1,j}$, one finds

$$a_{1,j} \left[\frac{j+1}{\xi} \frac{\eta_{1,j}^*}{\xi_0} + \frac{d}{d\xi} \left(\frac{\eta_{1,j}^*}{\xi_0} \right) \right]_{\xi_0} = \frac{(2j+1)}{\tau} \left(\frac{\xi_0}{\tau} \right)^{j-1} \quad (j = 2, 3, 4),$$

or, more exactly,

$$a_{1,j} = \frac{(2j+1) \left(\frac{\xi_0}{\tau} \right)^j}{\lim_{\xi \rightarrow \xi_0} \left[(j+1) \frac{\eta_{1,j}^*}{\xi_0} + \xi \frac{d}{d\xi} \left(\frac{\eta_{1,j}^*}{\xi_0} \right) \right]} \quad (j = 2, 3, 4) \quad (135)$$

and

$$a_{1,j} \equiv 0 \quad (j = 0, 1, 5, 6, \dots). \quad (136)$$

Let Δ_j be defined by

$$\Delta_j = \lim_{\xi \rightarrow \xi_0} \left[\frac{(2j+1) \frac{\eta_{1,j}^*}{\xi_0}}{(j+1) \frac{\eta_{1,j}^*}{\xi_0} + \xi \frac{d}{d\xi} \left(\frac{\eta_{1,j}^*}{\xi_0} \right)} \right] \quad (j = 2, 3, 4). \quad (137)$$

⁸ The validity of the relations (134) for $j = 0$ follows from eq. (131) and the convention that the potential shall vanish at infinity.

⁹ As in the rotational problem, this transformation is justified by the fact that the differential equations for $\eta_{1,j}$ are linear and homogeneous in $\eta_{1,j}$ (see eq. [140]).

Then, according to equation (135),

$$\Delta_j = a_{1,j} \left(\frac{\eta_{1,j}^*}{\xi_0} \right)_{\xi_0} \nu^{-j}, \quad (138)$$

where ν is

$$\nu = \frac{\xi_0}{\tau}. \quad (139)$$

Finally, eliminating $\chi'_{1,j}$ by combining (126) and (129) and using (136), one has as the equations to be solved in the tidal problem the set

$$\frac{1}{\xi^2} \left\{ \frac{d}{d\xi} \left(\xi^2 \frac{d}{d\xi} \left[\frac{\eta_{1,j}}{\xi_0} \right] \right) - j(j+1) \left[\frac{\eta_{1,j}}{\xi_0} \right] \right\} = -\eta_{1,j} \left(\frac{\frac{d\xi_0}{d\xi}}{\frac{d\eta_0}{d\xi}} \right) \quad (j=2,3,4). \quad (140)$$

The boundary $\xi = \Xi$ of the configuration is defined by the condition

$$\eta(\Xi) = 0 \quad (141)$$

or

$$\eta_0(\Xi) + \chi_0'' (\eta_{1,2} P_2 + \eta_{1,3} P_3 + \eta_{1,4} P_4) + \dots = 0. \quad (142)$$

Expanding each of the radial functions about the boundary $\xi = \xi_0$ of the corresponding unperturbed configuration, one has

$$\begin{aligned} \eta_0(\xi_0) + (\Xi - \xi_0) \left(\frac{d\eta_0}{d\xi} \right)_{\xi_0} + \dots \\ + \chi_0'' (\eta_{1,2} P_2 + \eta_{1,3} P_3 + \eta_{1,4} P_4)_{\xi_0} + \dots = 0 \end{aligned}$$

which, since $\eta_0(\xi_0) = 0$ gives for $(\Xi - \xi_0)$ the expression

$$\Xi - \xi_0 = - \frac{\chi_0''}{\left(\frac{d\eta_0}{d\xi} \right)_{\xi_0}} [\eta_{1,2} P_2 + \eta_{1,3} P_3 + \eta_{1,4} P_4]_{\xi_0} \quad (143)$$

to the first order in χ_0'' .

The mass of the star will be

$$\begin{aligned} M &= 2\pi \rho_c a^3 \int_0^{\Xi} \int_{-1}^1 \xi \xi^2 d\xi d\mu \\ &= 2\pi \rho_c a^3 \left\{ \int_0^{\xi_0} \int_{-1}^1 \xi \xi^2 d\xi d\mu + \int_{\xi_0}^{\Xi} \int_{-1}^1 \xi \xi^2 d\xi d\mu \right\}. \end{aligned} \quad (144)$$

Letting

$$\zeta(\xi) = \zeta(\xi_0) + (\xi - \xi_0) \left(\frac{d\zeta}{d\xi} \right)_{\xi_0} + \dots$$

in the second term of the right member, the mass, to the first order in χ_0'' , is

$$M = 4\pi \rho_c a^3 \int_0^{\xi_0} \xi_0 \xi^2 d\xi. \quad (145)$$

But by (120) and (122)

$$\xi_0 \xi^2 = - \frac{d}{d\xi} \left(\frac{\xi^2}{\xi_0} \frac{d\eta_0}{d\xi} \right), \quad (146)$$

so that

$$M = -4\pi\rho_c a^3 \left(\frac{\xi^2}{\xi_0} \frac{d\eta_0}{d\xi} \right)_{\xi_0}. \quad (147)$$

Using equation (147), we can write the expansion for η in an alternative form. By assumption,

$$\eta = \eta_0 + \chi_0'' [\eta_{1,2} P_2 + \eta_{1,3} P_3 + \eta_{1,4} P_4] + \dots;$$

but, by taking

$$\chi_0'' = \frac{M''}{4\pi\rho_c a^3 \tau} = - \left(\frac{M''}{M} \right) \xi_0 \left(\frac{\xi_0}{\tau} \right) \left[\frac{1}{\xi_0} \frac{d\eta_0}{d\xi} \right]_{\xi_0} = - \xi_0 \nu \left(\frac{M''}{M} \right) \left[\frac{1}{\xi_0} \frac{d\eta_0}{d\xi} \right]_{\xi_0}, \quad (148)$$

one has

$$\eta = \eta_0 - \left[\frac{\xi}{\xi_0} \frac{d\eta_0}{d\xi} \right]_{\xi_0} \left(\frac{M''}{M} \right) \sum_{j=2}^4 \Delta_j \nu^{j+1} \left(\frac{\eta_{1,j}^*}{\xi_0} \right)_{\xi_0}^{-1} \eta_{1,j}^* (\xi). \quad (149)$$

Similarly, the expression for the outer boundary is

$$\Xi = \xi_0 + \xi_0 \left(\frac{M''}{M} \right) \sum_{j=2}^4 \nu \left(\frac{\eta_{1,j}}{\xi_0} \right)_{\xi_0} P_j = \xi_0 + \xi_0 \left(\frac{M''}{M} \right) \sum_{j=2}^4 \nu^{j+1} \Delta_j P_j. \quad (150)$$

Let

$$\sigma(\mu) = \left| \frac{\Xi(\mu) - \xi_0}{\xi_0} \right|. \quad (151)$$

Then from (150)

$$\sigma(+1) = \frac{M''}{M} (\Delta_2 \nu^3 + \Delta_3 \nu^4 + \Delta_4 \nu^5), \quad (152)$$

$$\sigma(0) = \frac{M''}{M} \left(\frac{1}{2} \Delta_2 \nu^3 - \frac{3}{8} \Delta_4 \nu^5 \right), \quad (153)$$

$$\sigma(-1) = \frac{M''}{M} (\Delta_2 \nu^3 - \Delta_3 \nu^4 + \Delta_4 \nu^5). \quad (154)$$

The quantities $\sigma(+1)$ and $\sigma(-1)$ are the respective fractional elongations in the directions toward and away from the pole of the tidal force. The quantity $\sigma(0)$ is the fractional contraction in the plane normal to the pole of the tidal force.¹⁰

III. A CONFIGURATION SUBJECT TO SIMULTANEOUS TIDAL AND ROTATIONAL ACCELERATIONS

Consider a perfect compressible fluid mass subjected to the tidal accelerations and at the same time rotating about some *fixed, arbitrary* axis. In an equilibrium state the disposition of matter within the configuration is determined, as before, by the fundamental equations (1), (2), and (3), with

$$\mathfrak{B} = \mathfrak{B}' + \mathfrak{B}'' + \mathfrak{B}''', \quad (155)$$

¹⁰ For analysis indicating "furrow" see Chandrasekhar, *M.N.*, 93, 457, 1933. The results are identical.

where $\mathfrak{B}'$ is the gravitational potential function of the distorted fluid mass, $\mathfrak{B}''$ the tidal potential of the secondary, and $\mathfrak{B}'''$ the rotational potential function. Now the tidal effects of the secondary star can be derived from the potential function

$$\mathfrak{B}'' = \frac{GM''}{R} \left(1 + \sum_{j=2}^4 \left(\frac{r}{R} \right)^j P_j(\mu) \right), \quad (156)$$

assuming that the direction $\varphi = 0$ ($\mu = 1$) is along the line of centers, the positive sense being taken to be in the direction toward the primary. To be able to calculate the rota-

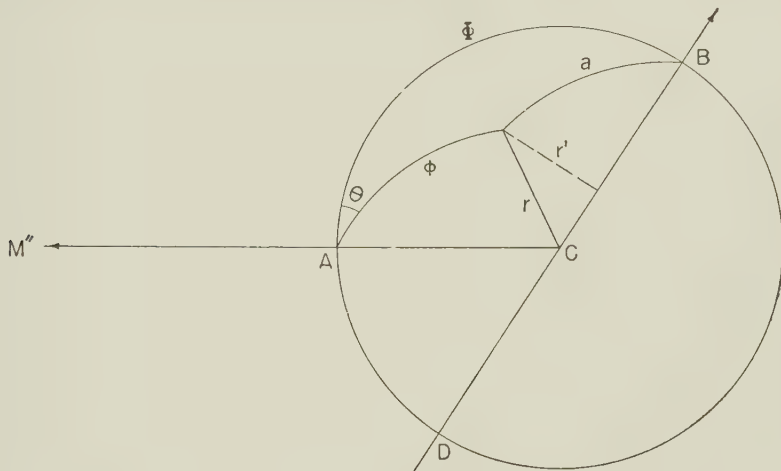


FIG. 1

tional potential, let the axis of rotation be taken to be the generator $\theta = 0$ of the cone $\varphi = \Phi$, and let the angular velocity about the axis DB be ω (=constant). Then the potential function $\mathfrak{B}'''$ will be

$$\mathfrak{B}''' = \frac{1}{2} \omega^2 r^2 \left[1 - \mu^2 \cos^2 \Phi - (1 - \mu^2) \cos^2 \theta \sin^2 \Phi - 2\mu \sqrt{1 - \mu^2} \cos \theta \sin \Phi \cos \Phi \right], \quad (157)$$

for

$$\mathfrak{B}''' = \frac{1}{2} \omega^2 (r')^2 = \frac{1}{2} \omega^2 r^2 \sin^2 a.$$

But by the cosine law of spherical trigonometry

$$\cos a = \mu \cos \Phi + \sqrt{1 - \mu^2} \cos \theta \sin \Phi,$$

whence

$$\sin^2 a = 1 - \cos^2 a = 1 - [\mu^2 \cos^2 \Phi + 2\mu \sqrt{1 - \mu^2} \cos \theta \sin \Phi \cos \Phi + (1 - \mu^2) \cos^2 \theta \sin^2 \Phi].$$

From this result, equation (157) follows at once. Setting

$$2 \sin \Phi \cos \Phi = a, \quad \cos^2 \Phi = b, \quad \sin^2 \Phi = c, \quad b + c = 1, \quad (158)$$

it becomes

$$\mathfrak{B}''' = \frac{1}{2} \omega^2 r^2 [1 - b\mu^2 - c(1 - \mu^2) \cos^2 \theta - a\mu \sqrt{1 - \mu^2} \cos \theta]. \quad (159)$$

For convenience, it is useful to express this in terms of the surface harmonics $S_j(\mu, \theta)$, which are solutions of the equations

$$\frac{\partial}{\partial \mu} \left((1-\mu^2) \frac{\partial S_j}{\partial \mu} \right) + \frac{1}{(1-\mu^2)} \frac{\partial^2 S_j}{\partial \theta^2} + j(j+1) S_j = 0 \quad (j = 0, 1, 2, \dots). \quad (160)$$

Now the $S_j(\mu, \theta)$ are of the form

$$S_j(\mu, \theta) = \sum_{m=0}^j (C_m^j \cos m\theta + D_m^j \sin m\theta) (1-\mu^2)^{m/2} \frac{d^m P_j}{d\mu^m}, \quad (161)$$

where the C_m^j and D_m^j are constants and the P_j the set of Legendre polynomials. It can be verified easily, by comparing (161) with (159), that

$$\mathfrak{B}''' = \frac{1}{2} \omega^2 r^2 \left[\frac{2}{3} + S_2(\mu, \theta) \right], \quad (162)$$

where in $S_2(\mu, \theta)$ the coefficients C_m^2 and D_m^2 have the values

$$C_0^2 = \frac{1}{3} (c - 2b), \quad C_1^2 = -\frac{1}{3} a, \quad C_2^2 = -\frac{1}{6} c, \quad D_m^2 = 0 \quad (m = 0, 1, 2). \quad (163)$$

Therefore, finally,

$$\mathfrak{B} = \mathfrak{B}' + \frac{GM''}{R} \left(1 + \sum_{j=2}^4 \left(\frac{r}{R} \right)^j P_j(\mu) \right) + \frac{1}{2} \omega^2 r^2 \left[\frac{2}{3} + S_2(\mu, \theta) \right]. \quad (164)$$

Using the dimensionless variables, the fundamental equations can be written in the form

$$\eta = \eta(\xi), \quad (165)$$

$$\frac{1}{\xi} \frac{\partial \eta}{\partial \xi} = \frac{\partial \chi'}{\partial \xi} + \chi_0'' \sum_{j=2}^4 \frac{j}{\tau} \left(\frac{\xi}{\tau} \right)^{j-1} P_j + \epsilon \xi \left[\frac{2}{3} + S_2 \right], \quad (166)$$

$$\frac{1}{\xi} \frac{\partial \eta}{\partial \mu} = \frac{\partial \chi'}{\partial \mu} + \chi_0'' \sum_{j=2}^4 \left(\frac{\xi}{\tau} \right)^j \frac{dP_j}{d\mu} + \frac{1}{2} \epsilon \xi^2 \frac{\partial S_2}{\partial \mu}, \quad (167)$$

$$\frac{1}{\xi} \frac{\partial \eta}{\partial \theta} = \frac{\partial \chi'}{\partial \theta} + \frac{1}{2} \epsilon \xi^2 \frac{\partial S_2}{\partial \theta}, \quad (168)$$

$$\frac{1}{\xi^2} \left\{ \frac{\partial}{\partial \xi} \left(\xi^2 \frac{\partial \chi'}{\partial \xi} \right) + \frac{\partial}{\partial \mu} \left((1-\mu^2) \frac{\partial \chi'}{\partial \mu} \right) + \frac{1}{1-\mu^2} \frac{\partial^2 \chi'}{\partial \theta^2} \right\} = -\xi, \quad (169)$$

$$\frac{1}{\xi^2} \left\{ \frac{\partial}{\partial \xi} \left(\xi^2 \frac{\partial \chi''}{\partial \xi} \right) + \frac{\partial}{\partial \mu} \left((1-\mu^2) \frac{\partial \chi''}{\partial \mu} \right) + \frac{1}{1-\mu^2} \frac{\partial^2 \chi''}{\partial \theta^2} \right\} = 0. \quad (170)$$

For small values of the parameters χ_0'' and ϵ one can seek solutions in the series forms

$$\eta(\xi, \mu, \theta; \chi_0'', \epsilon) = \eta_{0,0}(\xi) + \chi_0'' \eta_{1,0}(\xi, \mu, \theta) + \epsilon \eta_{0,1}(\xi, \mu, \theta) + \dots, \quad (171)$$

$$\xi(\xi, \mu, \theta; \chi_0'', \epsilon) = \xi_{0,0}(\xi) + \chi_0'' \xi_{1,0}(\xi, \mu, \theta) + \epsilon \xi_{0,1}(\xi, \mu, \theta) + \dots, \quad (172)$$

$$\chi'(\xi, \mu, \theta; \chi_0'', \epsilon) = \chi_{0,0}'(\xi) + \chi_0'' \chi_{1,0}'(\xi, \mu, \theta) + \epsilon \chi_{0,1}'(\xi, \mu, \theta) + \dots \quad (173)$$

Introducing these expressions into the fundamental equations and equating the coefficients of like powers of the parameters, one finds

$$\frac{1}{\zeta_{0,0}} \frac{d\eta_{0,0}}{d\xi} = \frac{d\chi'_{0,0}}{d\xi}, \quad (174)$$

$$\frac{1}{\zeta_{0,0}} \left\{ \frac{\partial \eta_{1,0}}{\partial \xi} - \frac{\zeta_{1,0}}{\zeta_{0,0}} \frac{d\eta_{0,0}}{d\xi} \right\} = \frac{\partial \chi'_{1,0}}{\partial \xi} + \sum_{j=2}^4 \frac{j}{\tau} \left(\frac{\xi}{\tau} \right)^{j-1} P_j, \quad (175)$$

$$\frac{1}{\zeta_{0,0}} \frac{\partial \eta_{1,0}}{\partial \mu} = \frac{\partial \chi'_{1,0}}{\partial \mu} + \sum_{j=2}^4 \left(\frac{\xi}{\tau} \right)^j \frac{dP_j}{d\mu}, \quad (176)$$

$$\frac{1}{\xi^2} \left\{ \frac{d}{d\xi} \left(\xi^2 \frac{\partial \chi'_{1,0}}{\partial \xi} \right) + \frac{\partial}{\partial \mu} \left((1-\mu^2) \frac{\partial \chi'_{1,0}}{\partial \mu} \right) \right\} = -\zeta_{1,0}, \quad (177)$$

$$\frac{1}{\zeta_{0,0}} \left\{ \frac{\partial \eta_{0,1}}{\partial \xi} - \frac{\zeta_{0,1}}{\zeta_{0,0}} \frac{d\eta_{0,0}}{d\xi} \right\} = \frac{\partial \chi'_{0,1}}{\partial \xi} + \xi \left[\frac{2}{3} + S_2 \right], \quad (178)$$

$$\frac{1}{\zeta_{0,0}} \frac{\partial \eta_{0,1}}{\partial \mu} = \frac{\partial \chi'_{0,1}}{\partial \mu} + \frac{1}{2} \xi^2 \frac{\partial S_2}{\partial \mu}, \quad (179)$$

$$\frac{1}{\zeta_{0,0}} \frac{\partial \eta_{0,1}}{\partial \theta} = \frac{\partial \chi'_{0,1}}{\partial \theta} + \frac{1}{2} \xi^2 \frac{\partial S_2}{\partial \theta}, \quad (180)$$

$$\frac{1}{\xi^2} \left\{ \frac{\partial}{\partial \xi} \left(\xi^2 \frac{\partial \chi'_{0,1}}{\partial \xi} \right) + \frac{\partial}{\partial \mu} \left((1-\mu^2) \frac{\partial \chi'_{0,1}}{\partial \mu} \right) + \frac{1}{1-\mu^2} \frac{\partial^2 \chi'_{0,1}}{\partial \theta^2} \right\} = -\zeta_{0,1}. \quad (181)$$

To separate the variables, one assumes solutions of the form

$$\left. \begin{aligned} \eta_{\gamma,\delta} &= \sum_{j=0}^{\infty} \eta_{\gamma,\delta;j}(\xi) Q_{\gamma,\delta;j}(\mu, \theta), \\ \zeta_{\gamma,\delta} &= \sum_{j=0}^{\infty} \zeta_{\gamma,\delta;j}(\xi) Q_{\gamma,\delta;j}(\mu, \theta), \\ \chi'_{\gamma,\delta} &= \sum_{j=0}^{\infty} \chi'_{\gamma,\delta;j}(\xi) Q_{\gamma,\delta;j}(\mu, \theta) \end{aligned} \right\} (\gamma, \delta) = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}, \quad (182)$$

where $Q_{1,0;j} = P_j(\mu)$, the j th Legendre polynomial, and $Q_{0,1;j} = S_j(\mu, \theta)$, the j th surface harmonic.

Introducing these into the equations (175)–(181) and equating the coefficients of like functions gives

$$\frac{1}{\zeta_{0,0}} \left\{ \frac{d\eta_{1,0;0}}{d\xi} - \frac{\zeta_{1,0;0}}{\zeta_{0,0}} \frac{d\eta_{0,0}}{d\xi} \right\} = \frac{d\chi'_{1,0;0}}{d\xi}, \quad (183)$$

$$\frac{\eta_{1,0;j}}{\zeta_{0,0}} = \chi'_{1,0;j} + \left(\frac{\xi}{\tau} \right)^j \quad (j = 2, 3, 4), \quad (184)$$

$$\frac{\eta_{1,0;j}}{\zeta_{0,0}} = \chi'_{1,0;j} \quad (j = 1, 5, 6, \dots), \quad (185)$$

$$\frac{1}{\xi^2} \left\{ \frac{d}{d\xi} \left(\xi^2 \frac{d\chi'_{1,0;j}}{d\xi} \right) - j(j+1) \chi'_{1,0;j} \right\} = -\zeta_{1,0;j} \quad (j=0, 1, 2, \dots), \quad (186)$$

$$\frac{1}{\zeta_{0,0}} \left\{ \frac{d\eta_{0,1;0}}{d\xi} - \frac{\zeta_{0,1;0}}{\zeta_{0,0}} \frac{d\eta_{0,0}}{d\xi} \right\} = \frac{d\chi'_{0,1;0}}{d\xi} + \frac{2}{3} \xi, \quad (187)$$

$$\frac{\eta_{0,1;2}}{\zeta_{0,0}} = \chi'_{0,1;2} + \frac{1}{2} \xi^2, \quad (188)$$

$$\frac{\eta_{0,1;j}}{\zeta_{0,0}} = \chi'_{0,1;j} \quad (j \neq 0, 2), \quad (189)$$

$$\frac{1}{\xi^2} \left\{ \frac{d}{d\xi} \left(\xi^2 \frac{d\chi'_{0,1;j}}{d\xi} \right) - j(j+1) \chi'_{0,1;j} \right\} = -\zeta_{0,1;j} \quad (j=0, 1, 2, \dots). \quad (190)$$

A comparison of equations (183)–(186) with the corresponding equations of the tidal problems shows them to be identical, while the same is true of a comparison of equations (187)–(190) with the corresponding equations of the first-order rotational problem.

Hence, the solution of the problem is in principle given (to the first order) by the solutions of the separate rotational and tidal problems. This is the generalization of a result known to be valid for polytropic configurations.

The mass is, of course,

$$M = \rho_c a^3 \int_0^\Xi \int_{-1}^1 \int_0^{2\pi} \zeta \xi^2 d\xi d\mu d\theta.$$

All terms explicitly containing θ or μ vanish after integration, so that

$$M = -4\pi \rho_c a^3 \left[\frac{\xi^2}{\zeta_{0,0}} \frac{d\eta_{0,0}}{d\xi} + \epsilon \xi^2 \left(-\frac{2}{3} \xi + \frac{d}{d\xi} \left[\frac{\eta_{0,1;0}}{\zeta_{0,0}} \right] \right) \right]_{\xi_0}, \quad (191)$$

as in the problem of rotation alone.

The boundary of the configuration is fixed by the condition that the pressure be zero there. This implies that

$$\eta(\Xi) = (\Xi - \xi_0) \left(\frac{d\eta_{0,0}}{d\xi} \right)_{\xi_0} + \epsilon \eta_{0,1}(\xi_0) + \chi''_{0,1} \eta_{1,0}(\xi_0) + \dots = 0,$$

whence it follows that, to the first order in ϵ and χ''_0 ,

$$\left. \begin{aligned} \Xi - \xi_0 &= -\frac{1}{\left(\frac{d\eta_{0,0}}{d\xi} \right)_{\xi_0}} \left[\epsilon \{ \eta_{0,1;0} + \eta_{0,1;2} S_2 \} + \chi''_{0,1} \{ \eta_{1,0;2} P_2 + \eta_{1,0;3} P_3 + \eta_{1,0;4} P_4 \} \right]_{\xi_0} \\ &= -\frac{\epsilon}{\left(\frac{d\eta_{0,0}}{d\xi} \right)_{\xi_0}} \left[\eta_{0,1;0} + a_{0,1;2} \eta_{0,1;2}^* S_2 \right]_{\xi_0} + \xi_0 \frac{M''}{M} \sum_{j=2}^4 \Delta_j p^{j+1} P_j. \end{aligned} \right\} \quad (192)$$

IV. APPLICATIONS OF THE THEORY

The results of the previous sections yield the differential equations (62), (63), (86), (87), (88), and (140) for the rotational and tidal perturbations. The equations for the first-order perturbations have been integrated numerically for three models due to Henrich.¹¹ The models used are: (I) a wholly convective model with $y_c = (p_r/p_\theta)_{r=0} =$

¹¹ *Ap. J.*, **93**, 483, 1941; *ibid.*, **96**, 106, 1942.

$(1 - \beta_c)/\beta_c = 0.01$, having a mass $M = 1.23M_\odot\mu_0^{-2}$ (μ_0 being the molecular weight of the stellar material); (II) a composite point-source model with $\gamma_c = 0.01$, $M = 1.51M_\odot\mu_0^{-2}$; (III) a composite point-source model with $\gamma_c = 0.25$, $M = 10.9M_\odot\mu_0^{-2}$. The numerical results are given for a variable in terms of which the integration was most conveniently performed, namely,

$$\psi_{1,j} = \xi \frac{\eta_{1,j}}{\xi_0} \quad (j = 0, 2, 3, 4). \quad (193)$$

The equations in $\psi_{1,j}$ to be integrated are easily shown to be

$$\frac{d^2\psi_{1,0}}{d\xi^2} = 2\xi - \psi_{1,0} \xi_0 \frac{\frac{d\xi_0}{d\xi}}{\frac{d\eta_0}{d\xi}}, \quad (194)$$

$$\frac{d^2\psi_{1,j}}{d\xi^2} = \psi_{1,j} \left(\frac{j(j+1)}{\xi^2} - \xi_0 \frac{\frac{d\xi_0}{d\xi}}{\frac{d\eta_0}{d\xi}} \right) \quad (j = 2, 3, 4). \quad (195)$$

For completeness, the following are two formulae of interest adapted from Henrich's papers, namely,

$$\eta_0 = e^{(32/3)(y-y_c)} \left(\frac{y}{y_c} \right)^{5/3} \left(\frac{1+y}{1+y_c} \right), \quad (196)$$

$$T = T_c e^{(8/3)(y-y_c)} \left(\frac{y}{y_c} \right)^{2/3}, \quad (197)$$

to which may be added the relations

$$\left. \begin{aligned} \eta_{1,j} &= \frac{\psi_{1,j}}{\xi} \xi_0, \\ \xi_{1,j} &= \frac{\psi_{1,j}}{\xi} \xi_0 \frac{\frac{d\xi_0}{d\xi}}{\frac{d\eta_0}{d\xi}} \end{aligned} \right\} \quad (198)$$

The principal results are contained in Table 1, which gives the Δ_j 's obtained for the different stellar models. The effective polytropic indices, n_{eff} , have been obtained from

TABLE 1

Model	Δ_2	Δ_3	Δ_4	$\rho_c/\bar{\rho}$	n_{eff}
I.....	1.27871	1.10201	1.05237	6.08	1.5205
II.....	1.01559	1.00354	1.00125	70.2	3.3063
III.....	1.004237	1.000746	1.000003	191	3.8486

Δ_2 by inverse interpolation.¹² The more detailed results are given for models I, II, and III in Tables 2, 3, and 4, respectively.

¹² See Chandrasekhar, *M.N.*, **93**, 567, 1933, Table VIII.

TABLE 2
WHOLLY CONVECTIVE MODEL
($y_c=0.01$, $\xi_0=5.8202$)

ξ	$\psi_{1,0}$	$\psi_{1,2}^*$	$\psi_{1,3}^*$	$\psi_{1,4}^*$	ζ_0	$\zeta_0 \frac{d\zeta_0}{d\xi}$	$y \times 10^3$
0.0	0.000000	0.000000	0.000000	0.000000	1.0000	0.60564	1.00000
0.2	0.002663	0.007986	0.001598	0.000320	0.9960	.60476	0.99625
0.4	0.021230	0.063559	0.025463	0.010195	0.9839	.60215	0.98512
0.6	0.071224	0.212672	0.128045	0.076996	0.9643	.59783	0.96682
0.8	0.167425	0.498097	0.400935	0.322002	0.9373	.59183	0.94170
1.0	0.32355	0.95804	0.96728	0.97319	0.9037	.58420	0.91024
1.2	0.55199	1.62497	1.97711	2.39336	0.8642	.57502	0.87306
1.4	0.86359	2.52474	3.60177	5.10266	0.8197	.56435	0.83084
1.6	1.2676	3.67606	6.02796	9.79486	0.7709	.55227	0.78435
1.8	1.7714	5.09018	9.45148	17.3473	0.7189	.53887	0.73439
2.0	2.3809	6.77106	14.0713	28.8245	0.6647	.52425	0.68183
2.2	3.1004	8.71568	20.0840	45.4773	0.6091	.50849	0.62753
2.4	3.9327	10.9148	27.6790	68.7363	0.5531	.49170	0.57236
2.6	4.8796	13.3536	37.0352	100.204	0.4975	.47395	0.51709
2.8	5.9419	16.0131	48.3176	141.642	0.4430	.45535	0.46251
3.0	7.1200	18.8709	61.6771	194.966	0.3904	.43594	0.40930
3.2	8.4139	21.9029	77.2493	262.231	0.3402	.41582	0.35811
3.4	9.8240	25.0839	95.1568	345.634	0.2928	.39503	0.30942
3.6	11.3509	28.3895	115.511	447.506	0.2486	.37362	0.26369
3.8	12.9962	31.7968	138.417	570.328	0.2079	.35161	0.22126
4.0	14.7629	35.2856	163.978	716.741	0.1708	.32898	0.18236
4.2	16.6555	38.8399	192.302	889.579	0.1375	.30571	0.14717
4.4	18.6807	42.4480	223.508	1091.906	0.1079	.28174	0.11576
4.6	20.8476	46.1047	257.740	1327.078	0.08195	.25690	0.08815
4.8	23.1686	49.8111	295.174	1598.828	0.05967	.23097	0.06431
5.0	25.6599	53.5771	336.036	1911.375	0.04091	.20355	0.04416
5.2	28.3429	57.4222	380.622	2269.598	0.02555	.17391	0.02761
5.4	31.2455	61.3790	429.330	2679.289	0.01352	.14063	0.01463
5.6	34.4057	65.4981	482.711	3147.594	0.00486	.09998	0.009485
5.8	37.8799	69.8628	541.603	3684.035	0.00013	.02965	0.000137
ξ_0	38.2505	70.3265	547.938	3742.717	0.00000	0.00000	0.000000
$\left(\frac{d\psi_{1,i}^*}{d\xi}\right)_{\xi_0}$		23.08127	315.5762	2927.286			

TABLE 3
COMPOSITE MODEL
($y_c=0.01$, $\xi_i=1.7039$, $\xi_0=14.0903$)

ξ	$\psi_{1,0}$	$\psi_{1,2}^*$	$\psi_{1,3}^*$	$\psi_{1,4}^*$	ξ_0	$\xi_0 \frac{\frac{d\xi_0}{d\xi}}{\frac{d\eta_0}{d\xi}}$	η_0
2.0.....	2.3795	6.7669	14.063	28.809	0.6619	0.5638	0.5144
2.4.....	3.9152	10.864	27.567	68.490	.5400	.5409	.3815
2.8.....	5.8806	15.829	47.863	140.51	.4216	.4843	.2699
3.2.....	8.2884	21.510	76.175	259.26	.3177	.4132	.1838
3.6.....	11.175	27.791	113.75	442.19	.2328	.3409	.1216
4.0.....	14.606	34.620	161.99	710.57	.1671	.2746	.07873
4.4.....	18.678	42.010	222.58	1090.32	.1183	.2172	.05022
4.8.....	23.510	50.027	297.55	1612.95	.08286	.1698	.03171
5.2.....	29.241	58.771	389.25	2316.40	.05770	.1316	.01992
5.6.....	36.020	68.364	500.43	3245.86	.04002	.1013	.01247
6.0.....	44.008	78.942	634.15	4454.59	.02768	.07784	.007802
6.4.....	53.367	90.641	793.82	6004.60	.01914	.05996	.004872
6.8.....	64.265	103.60	983.14	7967.19	.01322	.04541	.003039
7.2.....	76.871	117.95	1206.15	10424.0	.009096	.03466	.001893
7.6.....	91.353	133.83	1467.15	13467.2	.006276	.02664	.001179
8.0.....	107.88	151.37	1770.72	17200.5	.004303	.01979	.0007287
8.4.....	126.62	170.70	2121.79	21740.0	.002964	.01517	.0004469
8.8.....	147.75	191.93	2525.49	27213.7	.002019	.01163	.0002701
9.2.....	171.41	215.19	2987.14	33762.4	.001356	.008839	.0001608
9.6.....	197.78	240.58	3512.37	41541.1	.0008960	.006634	.00009337
10.0.....	227.01	268.23	4107.06	50719.2	.0005795	.004907	.00005294
10.4.....	259.26	298.23	4777.40	61481.8	.0003648	.003559	.00002887
10.8.....	294.69	330.72	5529.84	74029.4	.0002220	.002522	.00001506
11.2.....	333.46	365.80	6371.13	88579.8	.0001291	.001734	.00000743
11.6.....	375.72	403.57	7308.21	105366.9	.00007079	.001141	.00000339
12.0.....	421.63	444.15	8348.30	124642.0	.00003577	.000699	.00000138
12.4.....	471.32	487.64	9498.77	146674.7	.00001606	.000539	.00000049
12.8.....	524.96	534.15	10767.2	171752.9	.00000599	.000202	.00000013
13.2.....	582.68	583.77	12161.5	200182.0	.00000161	.000081	.00000002
13.6.....	644.61	636.60	13689.7	232287.0	.00000021	.000020	.00000000
14.0.....	710.90	692.72	15360.0	268412.0	.00000000	.000000	.00000000
ξ_0	726.48	705.86	15757.6	277160.1	0.00000000	0.000000	0.00000000
$\left(\frac{d\psi_{1,i}^*}{d\xi}\right)_{\xi_0}$		146.4422	4445.727	98131.165			

TABLE 4
COMPOSITE MODEL
($y_0=0.25$, $\xi_i=2.3889$, $\xi_0=19.9986$)

ξ	$\psi_{1,0}$	$\psi_{1,2}^*$	$\psi_{1,3}^*$	$\psi_{1,4}^*$	ξ_0	$\xi_0 \frac{d\xi_0}{d\xi}$	y	η_0
0.0.....	0.00000	0.00000	0.00000	0.00000	1.00000	0.6618	0.25000	
0.2.....	0.00266	0.00799	0.00160	0.00032	0.99560	.6603	.24963	
0.4.....	0.02122	0.06352	0.02545	0.01019	0.98251	.6559	.24853	
0.6.....	0.07116	0.21238	0.12791	0.07693	0.96115	.6487	.24671	
0.8.....	0.16715	0.49690	0.40018	0.32152	0.93208	.6388	.24416	
1.2.....	0.55014	1.61683	1.96948	2.38592	0.85422	.6115	.23699	
1.6.....	1.26115	3.64694	5.99162	9.74750	0.75687	.5756	.22716	
2.0.....	2.36603	6.69973	13.9604	28.6440	0.64897	.5333	.21488	
2.4.....	3.90763	10.7818	27.4328	68.2579	0.53894	.4867	0.20038	
3.0.....	7.07856	18.6101	61.0673	193.468	0.37783	.4375		0.2936
4.0.....	14.7777	34.9333	163.003	713.917	0.17557	.2813		.1060
5.0.....	26.4219	54.7364	342.523	1944.23	0.073370	.1534		.03486
6.0.....	44.0485	79.333	635.074	4456.69	0.029915	.07930		.01139
7.0.....	70.1879	110.855	1089.937	9122.10	0.012373	.04060		.003844
8.0.....	107.471	151.431	1768.314	17175.14	0.0052708	.02105		.001362
9.0.....	158.485	203.004	2741.834	30273.78	0.0023250	.01104		.0005049
10.0.....	225.746	267.373	4092.145	50558.85	0.0010716	.005776		.0001952
11.0.....	311.677	346.210	5910.257	80708.47	0.00050658	.003351		.00007503
12.0.....	418.558	441.055	8295.442	123982.0	0.00023328	.001913		.00002793
13.0.....	548.646	553.454	11356.9	184300.5	0.00010287	.001055		.00000986
14.0.....	704.139	684.899	15213.0	266289.1	0.00004239	.000553		.00000320
15.0.....	887.216	836.895	19992.2	375354.1	0.00001576	.000269		.00000092
16.0.....	1100.031	1010.973	25832.2	517742.7	0.00000499	.000118		.00000021
17.0.....	1344.732	1208.602	32880.8	700590.7	0.00000120	.000040		.00000002
18.0.....	1623.414	1431.304	41294.7	931992.9	0.00000017	.000010		.00000000
19.0.....	1938.095	1680.513	51238.7	1220991.3	0.00000001	.000001		.00000000
ξ_0	2290.255	1957.245	62869.2	1576768.8	0.00000000	0.000000		0.00000000
$\left(\frac{d\psi_{1,i}^*}{d\xi}\right)_{\xi_0}$		291.5425	12558.318	394218.0				

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